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Equilibrium of Elastic Nets

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Equilibrium of elastic nets

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A general equilibrium theory for nets constructed from two families of perfectly flexible elastic fibres is presented. The fibres are assumed to be continuously distributed and to offer negligible resistance to shear distortion. Configurations of nets are shown to be minimizers of the potential energy of deformation only if the associated fibre stretches are points of convexity of the fibre strain energy functions and the stresses in the fibres are tensile. These results are used to construct a relaxed energy density that automatically accounts for wrinkling of the network.

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Universal solutions are obtained. These are the deformations that can be maintained in every elastic net by the application of edge tractions and lateral pressure alone. A detailed study of the differential geometry of nets is included to aid in their interpretation.

The equilibrium theory for half-slack (wrinkled) nets is developed and applied to the solution of some representative examples.

1. Introduction

In this work we study the mechanics of elastic networks, modelled as membranes consisting of two families of perfectly flexible elastic fibres. The fibres are assumed to be continuously distributed and fastened together at their points of intersection to prevent slipping of one fibre family relative to the other. The model that we envision here is applicable to coarse mesh nets in which the force transmitted by a family of fibres depends only on the stretch of that family and the resistance to shear distortion (angle change between the fibres) is negligible.

A similar theory for nets consisting of inextensible fibres was proposed by Tchebychev (1878) and later used to study the problem of mapping a plane orthogonal network of inextensible fibres onto a surface whose shape is specified in advance. This is known as the problem of clothing a surface and is now a standard problem in differential geometry (Stoker 1969, §20). Recently, Pipkin (1980, 1981, 1984) refined this problem by accounting for elastic resistance to shear distortion while retaining the constraint of fibre inextensibility.

A theory for plane deformations of inextensible networks was formulated by Rivlin (1955) and Adkins (1956), and is summarized and extended in the book by Green & Adkins (1970, ch. 4). Rivlin (1959) has also presented a theory for curved surfaces, including an analysis of small deformations superposed on a finite deformation. Pipkin (1986*a*) and Pipkin & Rogers (1987) have extended the theory to account for the possibility of wrinkling of the network. Parallel developments in the mechanics of nets without shear resistance are reported in a series of papers by Kuznetsov (1965, 1969, 1982, 1984), whose work is primarily concerned with establishing existence of an equilibrated network on a given surface. The effects of fibre elasticity have recently been studied by Green & Shi (1989) in the context of a general theory for plane deformations. Wrinkling is taken into account but shear resistance is neglected. We develop this theory further in the present work and extend it to account for arbitrary deformations of curved surfaces.

In §2 we establish the notation and set down some elementary results from tensor analysis and differential geometry. A general equilibrium theory for membranes regarded as two-dimensional elastic surfaces is formulated in §3. The equilibrium equations are the Euler–Lagrange equations associated with the potential energies of edge loaded or pressurized membranes. A theory for networks consisting of elastic fibres that are initially orthogonal is developed in §4. This theory is obtained simply by specializing the strain energy of a general membrane to reflect our basic hypotheses regarding the mechanical behaviour of nets.

Section 5 is devoted to the differential geometry of nets. We obtain the Gauss and Mainardi–Codazzi equations of compatibility in forms that involve the fibre stretches, the shear distortion, the normal fibre curvatures and the torsion. Of particular importance here are the geodesic curvatures of the fibres and certain auxiliary parameters depending on the strain that are used to describe the intrinsic geometry of the net. Kuznetsov (1982) has referred to these parameters as Tchebychev curvatures in a less general context. We retain this terminology here.

Deformations that furnish local minima of the potential energy are studied in §6. We use the necessary condition of rank-one convexity (Graves 1939) to prove that in an energy minimizing configuration the fibre stresses are non-negative. Furthermore, the stretch of a family of fibres must belong to a domain of convexity of the strain energy function for that family. If these requirements are met and the net is in equilibrium under fixed edge tractions and boundary placements, then the deformation is a global minimizer of the potential energy.

Certain mild constitutive assumptions are introduced in §7 which, taken together with the results of §6, lead us to conclude that the energy minimization problem generally has no solution. This difficulty is resolved by using a so-called relaxed energy in place of the original strain energy function. Existence of solutions to the relaxed minimization problem has been established for a class of energy functionals containing those specific to the theory of elastic nets (Morrey 1952; Dacorogna 1989). We find that a relaxed fibre strain energy is a convex, non-decreasing function of the fibre stretch that vanishes if the stretch is compressive. For compressive stretches, the deformation of the net can be interpreted as the smooth limit of a sequence of finely wrinkled configurations in which the deformation gradients oscillate rapidly and discontinuously. We construct such a sequence to show how the relaxed strain energy can be obtained from the original energy. This idea has been used previously to derive a theory for wrinkled inextensible nets (Pipkin 1986*a*) and a general tension-field theory for wrinkled isotropic elastic membranes (Pipkin 1986*b*; Steigmann & Pipkin 1989; Steigmann 1990).

The relaxed strain energy is used as the basis for the equilibrium theory of §§8–11. In §8 we derive the equilibrium equations for nets in which both fibre families are tense, and for half-slack nets in which one family is tense and the other wrinkled. The problem of finding universal deformations of tense nets is addressed in §§9 and 10. These are the deformations that can be maintained in a net by application of edge traction and lateral pressure alone for all fibre strain energy functions. Those restrictions arising only from the tangential equations of equilibrium are obtained in §9. The Gauss and Mainardi–Codazzi conditions are used in §10 to find further restrictions necessary for equilibrium under uniform lateral pressure, including the special case of zero pressure.

The general equilibrium theory for half-slack nets is developed in §11. This theory is of such a simple form that the deformation can be obtained explicitly in the absence of lateral pressure. Furthermore, the traction problem is statically determinate in the sense that the fibre stress is described by an equation that does not involve the deformation. Finally, we specialize the theory to describe deformations of half-slack nets that initially lie in the plane or on general surfaces of revolution.

2. Summary of notation

Our analysis is based on convected Gauss coordinates θ^α $(\alpha = 1, 2)$ on the membrane surface which maintain a fixed one to one correspondence with material points under deformation. In this section relevant results and notations are recorded for later use (see Green & Zerna 1968).

A deformation carries the material point p with coordinates θ^α from its reference position $\boldsymbol{x}(\theta^1, \theta^2)$ to the place $\boldsymbol{r}(\theta^1, \theta^2)$ in euclidean 3-space, and induces the natural basis $\{\boldsymbol{a}_\alpha\}$,

$$\boldsymbol{a}_\alpha = \boldsymbol{r}_{,\alpha}, \tag{2.1}$$

which spans the tangent plane of the deformed surface at p, provided that $\boldsymbol{r}_{,1} \times \boldsymbol{r}_{,2} \neq \boldsymbol{0}$. Commas are used to denote partial derivatives with respect to the θ^α. Then

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{a}_\alpha \, \mathrm{d}\theta^\alpha \tag{2.2}$$

and the first fundamental form on the deformed surface is

$$|\mathrm{d}\boldsymbol{r}|^2 = a_{\alpha\beta} \, \mathrm{d}\theta^\alpha \, \mathrm{d}\theta^\beta, \tag{2.3}$$

where

$$a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \tag{2.4}$$

is the metric tensor. The determinant of the matrix $(a_{\alpha\beta})$ is denoted by $a = \det(a_{\alpha\beta})$ and is non-negative according to (2.3). When $a > 0$ the reciprocal metric components $a^{\alpha\beta}$ are uniquely defined by the relations

$$a^{\alpha\gamma} a_{\gamma\beta} = \delta^\alpha_\beta, \tag{2.5}$$

where δ^α_β is the Kronecker delta. These then give the reciprocal basis $\{\boldsymbol{a}^\alpha\}$ according to

$$\boldsymbol{a}^\alpha = a^{\alpha\beta} \boldsymbol{a}_\beta. \tag{2.6}$$

The directed unit normal $\boldsymbol{n}(\theta^1, \theta^2)$ to the deformed surface is given by

$$\epsilon_{\alpha\beta} \boldsymbol{n} = \boldsymbol{a}_\alpha \times \boldsymbol{a}_\beta, \quad \boldsymbol{n} = \tfrac{1}{2} \epsilon^{\alpha\beta} \boldsymbol{a}_\alpha \times \boldsymbol{a}_\beta, \tag{2.7}$$

where

$$\epsilon_{\alpha\beta} = a^{\frac{1}{2}} e_{\alpha\beta}, \quad \epsilon^{\alpha\beta} = a^{-\frac{1}{2}} e^{\alpha\beta} \tag{2.8}$$

and $e^{\alpha\beta} = e_{\alpha\beta}$ is the unit alternator, equal to 1 or -1 according as $(\alpha, \beta) = (1, 2)$, $(2, 1)$, respectively, and zero otherwise. The relations

$$\boldsymbol{n} \times \boldsymbol{a}^\alpha = \epsilon^{\alpha\beta} \boldsymbol{a}_\beta, \quad \boldsymbol{n} \times \boldsymbol{a}_\alpha = \epsilon_{\alpha\beta} \boldsymbol{a}^\beta \tag{2.9}$$

will also be used.

The curvature tensor and the Christoffel symbols are defined by

$$b_{\alpha\beta} = \boldsymbol{n} \cdot \boldsymbol{a}_{\alpha,\beta} = -\boldsymbol{a}_\alpha \cdot \boldsymbol{n}_{,\beta} \tag{2.10}$$

and

$$\Gamma^\gamma_{\alpha\beta} = \boldsymbol{a}^\gamma \cdot \boldsymbol{a}_{\alpha,\beta}, \tag{2.11}$$

respectively. Then

$$\boldsymbol{a}_{\alpha,\beta} = \Gamma^\gamma_{\alpha\beta} \boldsymbol{a}_\gamma + b_{\alpha\beta} \boldsymbol{n}. \tag{2.12}$$

Corresponding results for the reference surface may be deduced with the aid of notations

$$\left.\begin{array}{l} \boldsymbol{A}_\alpha = \boldsymbol{x}_{,\alpha}, \quad A_{\alpha\beta} = \boldsymbol{A}_\alpha \cdot \boldsymbol{A}_\beta, \quad A = \det(A_{\alpha\beta}), \quad A^{\alpha\gamma} A_{\gamma\beta} = \delta^\alpha_\beta, \\ \boldsymbol{A}^\alpha = A^{\alpha\beta} \boldsymbol{A}_\beta, \quad \mu_{\alpha\beta} = A^{\frac{1}{2}} e_{\alpha\beta}, \quad \mu^{\alpha\beta} = A^{-\frac{1}{2}} e^{\alpha\beta}, \quad \mu_{\alpha\beta} \boldsymbol{N} = \boldsymbol{A}_\alpha \times \boldsymbol{A}_\beta, \end{array}\right\} \tag{2.13}$$

and

$$B_{\alpha\beta} = \boldsymbol{N} \cdot \boldsymbol{A}_{\alpha,\beta} = -\boldsymbol{A}_\alpha \cdot \boldsymbol{N}_{,\beta}, \quad \bar{\Gamma}^\gamma_{\alpha\beta} = \boldsymbol{A}^\gamma \cdot \boldsymbol{A}_{\alpha,\beta}. \tag{2.14}$$

Let scalar functions $u_\alpha(\theta^1, \theta^2)$ and $u^\alpha(\theta^1, \theta^2)$ be related by $u^\alpha = A^{\alpha\beta} u_\beta$. These induce a surface vector

$$\boldsymbol{u} = u^\alpha \boldsymbol{A}_\alpha = u_\alpha \boldsymbol{A}^\alpha, \tag{2.15}$$

with partial derivatives $\boldsymbol{u}_{,\alpha}$ given by

$$\boldsymbol{u}_{,\alpha} - u^\beta B_{\beta\alpha} \boldsymbol{N} = u_{\beta|\alpha} \boldsymbol{A}^\beta = u^\beta{}_{|\alpha} \boldsymbol{A}_\beta, \tag{2.16}$$

where

$$u^\beta{}_{|\alpha} = u^\beta{}_{,\alpha} + u^\gamma \bar{\Gamma}^\beta_{\alpha\gamma}, \quad u_{\beta|\alpha} = u_{\beta,\alpha} - u_\gamma \bar{\Gamma}^\gamma_{\beta\alpha} \tag{2.17}$$

are covariant derivatives on the reference surface.

Let $\boldsymbol{L}$ and $\boldsymbol{M}$ be orthogonal unit vectors spanning the tangent plane of the reference surface at the point p. Then

$$\boldsymbol{L} = L_\alpha \boldsymbol{A}^\alpha = L^\alpha \boldsymbol{A}_\alpha, \quad \boldsymbol{M} = M_\alpha \boldsymbol{A}^\alpha = M^\alpha \boldsymbol{A}_\alpha \tag{2.18}$$

and

$$\left.\begin{aligned} \boldsymbol{A}_\alpha &= L_\alpha \boldsymbol{L} + M_\alpha \boldsymbol{M}, \quad \boldsymbol{A}^\alpha = L^\alpha \boldsymbol{L} + M^\alpha \boldsymbol{M}, \\ A_{\alpha\beta} &= L_\alpha L_\beta + M_\alpha M_\beta, \quad A^{\alpha\beta} = L^\alpha L^\beta + M^\alpha M^\beta. \end{aligned}\right\} \tag{2.19}$$

If $\boldsymbol{L}$ and $\boldsymbol{M}$ are oriented so that $\boldsymbol{L} \times \boldsymbol{M} = \boldsymbol{N}$, then we conclude from (2.13, 19) that

$$\mu_{\alpha\beta} = L_\alpha M_\beta - M_\alpha L_\beta, \quad \mu^{\alpha\beta} = L^\alpha M^\beta - M^\alpha L^\beta. \tag{2.20}$$

We use a version of the Green–Stokes theorem,

$$\iint_D v^\alpha{}_{,\alpha}\, \mathrm{d}\theta^1\, \mathrm{d}\theta^2 = \oint_{\partial D} v^\alpha e_{\alpha\beta}\, \mathrm{d}\theta^\beta, \tag{2.21}$$

for smooth functions $v^\alpha(\theta^1, \theta^2)$ and arbitrary regular regions D of the (θ^1, θ^2)-plane: Let $v^{i\alpha}(\theta^1, \theta^2)$ $(i = 1, 2, 3;\ \alpha = 1, 2)$ be smooth and let $\{\boldsymbol{e}_i\}$ be a fixed rectangular basis. For fixed i, apply (2.21) to $v^{i\alpha}$, multiply by $\boldsymbol{e}_i$ and sum to get

$$\iint_D \boldsymbol{v}^\alpha{}_{,\alpha}\, \mathrm{d}\theta^1\, \mathrm{d}\theta^2 = \oint_{\partial D} \boldsymbol{v}^\alpha e_{\alpha\beta}\, \mathrm{d}\theta^\beta, \tag{2.22}$$

where $\boldsymbol{v}^\alpha = v^{i\alpha} \boldsymbol{e}_i$.

3. Membrane theory

In this work we treat nets as elastic membranes with special properties, to be specified in §4. In the present section we outline the membrane theory of elastic surfaces as developed by Stoker (1964) and Green *et al.* (1965). Thus we regard the membrane as a two-dimensional elastic continuum with a strain energy W, measured per unit area of the reference surface. The deformation gradient $\boldsymbol{F}$ is the mapping defined by

$$\mathrm{d}\boldsymbol{r} = \boldsymbol{F}\, \mathrm{d}\boldsymbol{x}. \tag{3.1}$$

W is taken to be a function of $\boldsymbol{F}$ and the particle p with coordinates (θ^1, θ^2). However, we do not indicate dependence on the latter variable explicitly. By using (2.2) and $\mathrm{d}\theta^\alpha = \boldsymbol{A}^\alpha \cdot \mathrm{d}\boldsymbol{x}$ in (3.1), we obtain

$$\boldsymbol{F} = \boldsymbol{a}_\alpha \otimes \boldsymbol{A}^\alpha; \quad \boldsymbol{a}_\alpha = \boldsymbol{F}\boldsymbol{A}_\alpha. \tag{3.2}$$

Let $r^k (k = 1, 2, 3)$ be the components of $\boldsymbol{r}$ on a fixed rectangular basis $\{\boldsymbol{e}_k\}$. Then the Piola stress $\boldsymbol{T}$, measuring forces in the membrane per unit length of arc on the reference surface, is defined by

$$\boldsymbol{T} = \boldsymbol{T}^\alpha \otimes \boldsymbol{A}_\alpha; \quad \boldsymbol{T}^\alpha = (\partial W / \partial r^i{}_{,\alpha})\, \boldsymbol{e}_i. \tag{3.3}$$

We assume that W is frame indifferent, i.e. $W(\boldsymbol{F}) = W(\boldsymbol{Q}\boldsymbol{F})$ for all proper orthogonal $\boldsymbol{Q}$. It follows that $W(\boldsymbol{F}) = \hat{W}(\boldsymbol{C})$, where

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} = a_{\alpha\beta} \boldsymbol{A}^\alpha \otimes \boldsymbol{A}^\beta \tag{3.4}$$

is the Cauchy–Green strain (see Cohen & Wang 1984). If $\hat{W}$ is symmetrized in the $a_{\alpha\beta}$, we find

$$\boldsymbol{T}^\alpha = J\sigma^{\alpha\beta} \boldsymbol{a}_\beta, \tag{3.5}$$

where
$$J = (a/A)^{\frac{1}{2}} \tag{3.6}$$
is the areal stretch and
$$\sigma^{\alpha\beta} = 2J^{-1}\partial\hat{W}/\partial a_{\alpha\beta} \tag{3.7}$$
are the contravariant components of the Cauchy stress
$$\boldsymbol{\sigma} = \sigma^{\alpha\beta}\boldsymbol{a}_\alpha \otimes \boldsymbol{a}_\beta. \tag{3.8}$$

Suppose the domain of the function $\boldsymbol{x}(\cdot)$ is an open region S of the (θ^1, θ^2)-plane with piecewise smooth boundary ∂S. Let the membrane be subjected to loads derivable from a potential $P[\boldsymbol{r}]$. Then we define a potential energy $E[\boldsymbol{r}]$ of the deformation by
$$E[\boldsymbol{r}] = \iint_S W(\boldsymbol{F})A^{\frac{1}{2}}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2 - P[\boldsymbol{r}]. \tag{3.9}$$
We consider two types of loading.

(i) *Pressure loading*

In this case the membrane is a closed surface subjected to a spatially uniform, volume-dependent pressure p acting on the interior surface. Then
$$P[\boldsymbol{r}] = \int_{V_0}^{V[\boldsymbol{r}]} p(v)\,\mathrm{d}v, \tag{3.10}$$
where
$$V[\boldsymbol{r}] = \iint_S \chi(\boldsymbol{r};\boldsymbol{r}_{,\alpha})\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2; \quad \chi = \tfrac{1}{6}e^{\alpha\beta}\boldsymbol{r}\cdot\boldsymbol{r}_{,\alpha}\times\boldsymbol{r}_{,\beta}. \tag{3.11}$$
Here $V[\boldsymbol{r}]$ is the volume enclosed by the deformed membrane and V_0 is an arbitrary constant.

(ii) *Dead loading*

The membrane is subjected to an edge load $\boldsymbol{t}$ applied to a part ∂S_t of the boundary. Here $\boldsymbol{t}$ represents force measured per unit arc length s of the curve $\boldsymbol{x}(\partial S_t)$. Let ∂S_r be the complement of ∂S_t and suppose $\boldsymbol{r}(\partial S_r)$ is prescribed. If $\boldsymbol{t}$ is assigned as a dead load then
$$P[\boldsymbol{r}] = \int_{\boldsymbol{x}(\partial S_t)} \boldsymbol{t}\cdot\boldsymbol{r}\,\mathrm{d}s = \int_{\partial S_t} \boldsymbol{t}\cdot\boldsymbol{r}s_\alpha\,\mathrm{d}\theta^\alpha, \tag{3.12}$$
where $\boldsymbol{s} = s_\alpha\boldsymbol{A}^\alpha$ is the unit tangent to the curve $\boldsymbol{x}(\partial S_t)$ and $s_\alpha = A_{\alpha\beta}\,\mathrm{d}\theta^\beta/\mathrm{d}s$.

The equilibrium equations for the membrane are the Euler–Lagrange equations associated with the functional E. For the problem of pressure loading these are (Stoker 1964; Steigmann 1990)
$$A^{-\frac{1}{2}}(A^{\frac{1}{2}}\boldsymbol{T}^\alpha)_{,\alpha} + pJ\boldsymbol{n} = \boldsymbol{0}; \quad (\theta^1,\theta^2)\in S. \tag{3.13}$$
These equations also describe the problem of frictionless contact of a membrane with a smooth surface. The contact pressure $p(\theta^1,\theta^2)$ is determined in the course of the analysis. For the dead-load problem, (3.13) remains valid with $p = 0$, and is supplemented by the natural boundary conditions
$$\boldsymbol{T}^\alpha\nu_\alpha = \boldsymbol{t}; \quad (\theta^1,\theta^2)\in\partial S_t, \tag{3.14}$$
where $\boldsymbol{\nu} = \nu_\alpha\boldsymbol{A}^\alpha$ is the rightward unit normal to the arc $\boldsymbol{x}(\partial S_t)$:
$$\boldsymbol{\nu} = \boldsymbol{s}\times\boldsymbol{N}; \quad \nu_\alpha = \mu_{\alpha\beta}\,\mathrm{d}\theta^\beta/\mathrm{d}s. \tag{3.15}$$

4. Elastic networks

We assume that the material fibres are arranged in an orthogonal network on the reference surface. The trajectories of this network are defined at each point p by a non-holonomic field of unit vectors $\boldsymbol{L}$ and $\boldsymbol{M}$, tangential to the fibres and oriented so that $\boldsymbol{L} \times \boldsymbol{M} = \boldsymbol{N}$. Thus equations (2.18)–(2.20) are applicable.

The deformation gradient $\boldsymbol{F}$ maps the material vectors $\boldsymbol{L}$ and $\boldsymbol{M}$ onto

$$\lambda \boldsymbol{l} = \boldsymbol{F}\boldsymbol{L}, \quad \mu \boldsymbol{m} = \boldsymbol{F}\boldsymbol{M}, \tag{4.1}$$

respectively, where

$$\lambda = |\boldsymbol{F}\boldsymbol{L}|, \quad \mu = |\boldsymbol{F}\boldsymbol{M}| \tag{4.2}$$

are the fibre stretches and $\boldsymbol{l}$ and $\boldsymbol{m}$ are unit tangents to the fibres on the deformed surface. From (2.19b) and (3.2a) it follows that

$$\boldsymbol{F} = L^{\alpha}\boldsymbol{a}_{\alpha} \otimes \boldsymbol{L} + M^{\alpha}\boldsymbol{a}_{\alpha} \otimes \boldsymbol{M}. \tag{4.3}$$

Then (2.1), (4.1) and (4.3) result in

$$\lambda \boldsymbol{l} = (\boldsymbol{L}\cdot\nabla)\,\boldsymbol{r}, \quad \mu \boldsymbol{m} = (\boldsymbol{M}\cdot\nabla)\,\boldsymbol{r}, \tag{4.4}$$

where $\nabla = \boldsymbol{A}^{\alpha}(\cdot)_{,\alpha}$ is the tangential gradient operator on the reference surface.

From the representations

$$\boldsymbol{l} = l^{\alpha}\boldsymbol{a}_{\alpha}, \quad \boldsymbol{m} = m^{\alpha}\boldsymbol{a}_{\alpha} \tag{4.5}$$

and (4.1) and (4.3), we find

$$\lambda l^{\alpha} = L^{\alpha}, \quad \mu m^{\alpha} = M^{\alpha}. \tag{4.6}$$

Thus (4.3), (4.5) and (4.6) yield

$$\boldsymbol{F} = \lambda \boldsymbol{l} \otimes \boldsymbol{L} + \mu \boldsymbol{m} \otimes \boldsymbol{M}. \tag{4.7}$$

The fibre shear angle γ is defined by

$$\sin\gamma = \boldsymbol{l}\cdot\boldsymbol{m}. \tag{4.8}$$

Then the Cauchy–Green strain is

$$\boldsymbol{C} = \lambda^2 \boldsymbol{L} \otimes \boldsymbol{L} + \mu^2 \boldsymbol{M} \otimes \boldsymbol{M} + \lambda\mu \sin\gamma(\boldsymbol{L} \otimes \boldsymbol{M} + \boldsymbol{M} \otimes \boldsymbol{L}), \tag{4.9}$$

and this furnishes the metric on the deformed surface in terms of the initial fibre directions, the stretches and the shear angle:

$$a_{\alpha\beta} = \lambda^2 L_{\alpha} L_{\beta} + \mu^2 M_{\alpha} M_{\beta} + \lambda\mu \sin\gamma (L_{\alpha} M_{\beta} + M_{\alpha} L_{\beta}). \tag{4.10}$$

This result will prove useful in our study of the differential geometry of nets in §5.

The square of the areal stretch is given by

$$J^2 = a/A = \tfrac{1}{2}\mu^{\alpha\beta}\mu^{\lambda\gamma} a_{\alpha\lambda}\, a_{\beta\gamma}. \tag{4.11}$$

By substituting (2.20b) and (4.10) into this we find

$$J = \lambda\mu|\cos\gamma|. \tag{4.12}$$

According to (2.19a), (3.2b) and (4.7), the tangent basis vectors $\boldsymbol{a}_{\alpha}$ on the deformed surface can be expressed as

$$\boldsymbol{a}_{\alpha} = \lambda \boldsymbol{l} L_{\alpha} + \mu \boldsymbol{m} M_{\alpha}. \tag{4.13}$$

Then (2.7 a), (2.20 a), (4.12) and (4.13) result in

$$\boldsymbol{l}\times\boldsymbol{m} = |\cos\gamma|\boldsymbol{n}, \tag{4.14}$$

which specifies the orientation of the deformed tangent plane in terms of the fibre directions.

We define an elastic net to be an orthotropic elastic membrane with particular properties. The material symmetry axes are taken to be the initial fibre directions $\boldsymbol{L}$ and $\boldsymbol{M}$. Thus the strain energy function $\hat{W}(\boldsymbol{C})$ is subject to the restrictions (Cohen & Wang 1984)

$$\hat{W}(\boldsymbol{C}) = \hat{W}(\boldsymbol{P}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{P}), \quad \boldsymbol{P}\in SG, \tag{4.15}$$

for all Cauchy–Green strains $\boldsymbol{C}$, where SG is the symmetry group:

$$SG = \{\pm(\boldsymbol{L}\otimes\boldsymbol{L}\pm\boldsymbol{M}\otimes\boldsymbol{M})\}. \tag{4.16}$$

From (4.9), (4.15) and (4.16) it is straightforward to show that the strain energy is expressible as an even function of λ, μ and $\sin\gamma$. Since λ and μ are non-negative by definition, we have $\hat{W}(\boldsymbol{C}) = w(\lambda,\mu,|\sin\gamma|)$.

Proceeding with our characterization of nets, we postulate that $w(\lambda,\mu,\cdot) = \hat{w}(\lambda,\mu)$ for all fibre stretches λ and μ, so that the local response of the membrane is unaffected by shearing of the fibres. Next, consider a unit square of homogeneous material whose composition is identical to that of a fixed point p of the reference surface. Let the edges of the square be parallel to the fibres. The forces required to deform the square into a rectangle of dimensions λ and μ are $\hat{w}_\lambda$ and $\hat{w}_\mu$. We postulate that the force carried by a family of fibres is independent of the stretch of the orthogonal family for all fibre stretches λ and μ, i.e. $\hat{w}_{\lambda\mu}\equiv 0$. Thus we conclude that the strain energy of an elastic net can be expressed in the form

$$\hat{W}(\boldsymbol{C}) = F(\lambda)+G(\mu), \tag{4.17}$$

where
$$\lambda^2 = \boldsymbol{L}\cdot\boldsymbol{C}\boldsymbol{L} = a_{\alpha\beta}L^\alpha L^\beta, \quad \mu^2 = \boldsymbol{M}\cdot\boldsymbol{C}\boldsymbol{M} = a_{\alpha\beta}M^\alpha M^\beta. \tag{4.18}$$

The stress is obtained from (3.7), (4.17) and (4.18):

$$J\sigma^{\alpha\beta} = \lambda^{-1}f(\lambda)\,L^\alpha L^\beta + \mu^{-1}g(\mu)\,M^\alpha M^\beta, \tag{4.19}$$

where
$$f(\lambda) = F'(\lambda) \quad\text{and}\quad g(\mu) = G'(\mu) \tag{4.20}$$

are the fibre stresses. We assume that $f(\cdot)$ and $g(\cdot)$ are continuous. From (3.5) and (4.5, 6) we get

$$\boldsymbol{T}^\alpha = f(\lambda)\,\boldsymbol{l}L^\alpha + g(\mu)\,\boldsymbol{m}M^\alpha. \tag{4.21}$$

Then the Piola stress is

$$\boldsymbol{T} = f(\lambda)\,\boldsymbol{l}\otimes\boldsymbol{L} + g(\mu)\,\boldsymbol{m}\otimes\boldsymbol{M}. \tag{4.22}$$

We further assume that

$$F(1) = G(1) = 0, \quad f(1) = g(1) = 0. \tag{4.23}$$

Then from (2.19 c), (4.10), (4.17) and (4.22) it is evident that the reference surface can be mapped with $\hat{W} = 0$ and $\boldsymbol{T} = \boldsymbol{0}$ onto any surface whose metric can be expressed in the form

$$a_{\alpha\beta} = A_{\alpha\beta} + \sin\gamma(L_\alpha M_\beta + M_\alpha L_\beta). \tag{4.24}$$

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Accordingly, any such surface may be regarded as a stress-free reference configuration for the elastic net.

5. Differential geometry of a network

To describe the deformed configurations of networks it is helpful to relate the deformation to the variables used in the classical differential geometry of surfaces (Struik 1961; Kreyszig 1968), such as the gaussian and mean curvatures and the intrinsic and extrinsic curvatures of the fibres. For our later work it is important to have formulae for these variables that involve the fibre stretches, the angle of shear and the initial fibre trajectories. It is well known that the intrinsic and extrinsic geometries of a surface are not independent, but are connected by the Gauss and Mainardi–Codazzi equations of compatibility. These relations aid in the interpretation of the universal solutions of §§9 and 10 when an explicit expression for the deformation is not available.

(a) *Curvature tensor – gaussian and mean curvatures*

We begin by deriving an expression for the covariant components $b_{\alpha\beta}$ of the curvature tensor in terms of the normal fibre curvatures

$$\kappa_l = b_{\alpha\beta} l^\alpha l^\beta, \quad \kappa_m = b_{\alpha\beta} m^\alpha m^\beta \tag{5.1}$$

and the torsion

$$\tau = b_{\alpha\beta} l^\alpha m^\beta. \tag{5.2}$$

To this end we observe from (4.6) that

$$b_{\alpha\beta} L^\alpha L^\beta = \lambda^2 \kappa_l, \quad b_{\alpha\beta} M^\alpha M^\beta = \mu^2 \kappa_m, \quad b_{\alpha\beta} L^\alpha M^\beta = \lambda\mu\tau. \tag{5.3}$$

We can use the components $b_{\alpha\beta}$ to define a tensor $\hat{\boldsymbol{b}} = b_{\alpha\beta} \boldsymbol{A}^\alpha \otimes \boldsymbol{A}^\beta$ on the reference surface. Substituting (2.19*b*) and (5.3), we find

$$\hat{\boldsymbol{b}} = \lambda^2 \kappa_l \boldsymbol{L} \otimes \boldsymbol{L} + \mu^2 \kappa_m \boldsymbol{M} \otimes \boldsymbol{M} + \lambda\mu\tau(\boldsymbol{L} \otimes \boldsymbol{M} + \boldsymbol{M} \otimes \boldsymbol{L}), \tag{5.4}$$

from which we recover the desired result:

$$b_{\alpha\beta} = \lambda^2 \kappa_l L_\alpha L_\beta + \mu^2 \kappa_m M_\alpha M_\beta + \lambda\mu\tau(L_\alpha M_\beta + M_\alpha L_\beta). \tag{5.5}$$

Now define $b = \det(b_{\alpha\beta})$ and recall that $A = \det(A_{\alpha\beta})$. Then

$$b/A = \tfrac{1}{2}\mu^{\alpha\beta}\mu^{\lambda\gamma} b_{\alpha\lambda} b_{\beta\gamma}, \tag{5.6}$$

and by substituting (2.20*b*) and (5.5) we obtain

$$b/A = (\lambda\mu)^2(\kappa_l \kappa_m - \tau^2). \tag{5.7}$$

This result, combined with (4.11) and (4.12), furnishes the gaussian curvature $\kappa = b/a$:

$$\kappa = \sec^2\gamma(\kappa_l \kappa_m - \tau^2). \tag{5.8}$$

Next we obtain an expression for the mean curvature $H = \tfrac{1}{2}a^{\alpha\beta} b_{\alpha\beta}$. From (4.10) it is easily confirmed that

$$J^2 a^{\alpha\beta} = \mu^2 L^\alpha L^\beta + \lambda^2 M^\alpha M^\beta - \lambda\mu \sin\gamma(L^\alpha M^\beta + M^\alpha L^\beta). \tag{5.9}$$

Then (4.12) yields

$$2H = \sec^2\gamma(\kappa_l + \kappa_m - 2\tau\sin\gamma). \tag{5.10}$$

Equations (5.8) and (5.10) are identical to formulae obtained elsewhere (Pipkin 1984) by a different method.

(b) Geodesic and Tchebychev curvatures

We introduce auxiliary unit vectors

$$\boldsymbol{p}=\boldsymbol{n}\times\boldsymbol{l}, \quad \boldsymbol{q}=\boldsymbol{n}\times\boldsymbol{m} \tag{5.11}$$

lying in the tangent plane of the deformed membrane. Then $\{\boldsymbol{l},\boldsymbol{p},\boldsymbol{n}\}$ and $\{\boldsymbol{m},\boldsymbol{q},\boldsymbol{n}\}$ are right-handed orthonormal bases for 3-space. We will have need of the gradients of the unit fibre vectors $\boldsymbol{l}$ and $\boldsymbol{m}$:

$$\boldsymbol{l}_{,\alpha}=x_\alpha\boldsymbol{p}+u_\alpha\boldsymbol{n}; \quad \boldsymbol{m}_{,\alpha}=y_\alpha\boldsymbol{q}+v_\alpha\boldsymbol{n}, \tag{5.12}$$

where $$x_\alpha=\boldsymbol{p}\cdot\boldsymbol{l}_{,\alpha}, \quad y_\alpha=\boldsymbol{q}\cdot\boldsymbol{m}_{,\alpha}, \quad u_\alpha=b_{\alpha\beta}l^\beta, \quad v_\alpha=b_{\alpha\beta}m^\beta. \tag{5.13}$$

One of our main objectives in the present section is to establish formulae for the tangential components of the directional derivatives

$$\left.\begin{aligned} l^\alpha\boldsymbol{l}_{,\alpha}&=\eta_l\boldsymbol{p}+\kappa_l\boldsymbol{n}, \quad m^\alpha\boldsymbol{m}_{,\alpha}=\eta_m\boldsymbol{q}+\kappa_m\boldsymbol{n},\\ m^\alpha\boldsymbol{l}_{,\alpha}&=\phi_l\boldsymbol{p}+\tau\boldsymbol{n}, \quad l^\alpha\boldsymbol{m}_{,\alpha}=\phi_m\boldsymbol{q}+\tau\boldsymbol{n},\end{aligned}\right\} \tag{5.14}$$

in terms of the fibre stretches λ and μ, the fibre shear angle γ and the initial fibre vectors $\boldsymbol{L}$ and $\boldsymbol{M}$. Here η_l and η_m are the geodesic curvatures of the $\boldsymbol{l}$- and $\boldsymbol{m}$-trajectories, respectively. The coefficient ϕ_l measures the tangential part of the rate of change of $\boldsymbol{l}$ with respect to arc length along an $\boldsymbol{m}$-fibre. ϕ_m has a similar interpretation. These latter variables are not discussed in the standard differential geometry texts. For axisymmetric configurations of nets they are equivalent to parameters referred to as Tchebychev curvatures by Kuznetsov (1982). We are unable to find any independent corroboration of this attribution. However, we will adhere to Kuznetsov's terminology and henceforth designate ϕ_l and ϕ_m as the Tchebychev curvatures of the $\boldsymbol{l}$- and $\boldsymbol{m}$-fibres, respectively.

According to (5.11 a) and (5.14 a),

$$\boldsymbol{n}\times l^\alpha\boldsymbol{l}\cdot\boldsymbol{l}_{,\alpha}=\eta_l \quad \text{and} \quad \boldsymbol{n}\times\boldsymbol{p}\cdot\boldsymbol{l}_{,\alpha}=0. \tag{5.15}$$

Multiplying the second equation by p^α $(=\boldsymbol{p}\cdot\boldsymbol{a}^\alpha)$, adding the result to the first equation and writing

$$\boldsymbol{a}^\alpha=p^\alpha\boldsymbol{p}+l^\alpha\boldsymbol{l} \tag{5.16}$$

leads to $$\eta_l=a^{-\frac{1}{2}}e^{\alpha\beta}l_{\beta,\alpha}. \tag{5.17}$$

Similarly we find

$$\eta_m=a^{-\frac{1}{2}}e^{\alpha\beta}m_{\beta,\alpha}. \tag{5.18}$$

We can write these in terms of covariant derivatives on the reference surface:

$$\eta_l=J^{-1}(\mu^{\alpha\beta}l_\beta)|_\alpha, \quad \eta_m=J^{-1}(\mu^{\alpha\beta}m_\beta)|_\alpha. \tag{5.19}$$

Now we use the relations $l_\alpha=a_{\alpha\beta}l^\beta$ and $m_\alpha=a_{\alpha\beta}m^\beta$ together with (4.6) and (4.10) to get

$$l_\alpha=\lambda L_\alpha+\mu\sin\gamma M_\alpha, \quad m_\alpha=\mu M_\alpha+\lambda\sin\gamma L_\alpha. \tag{5.20}$$

From these results and (2.20 b) we finally derive

$$J\eta_l=(\mu\sin\gamma L^\alpha-\lambda M^\alpha)|_\alpha, \quad J\eta_m=(\mu L^\alpha-\lambda\sin\gamma M^\alpha)|_\alpha, \tag{5.21}$$

where J is given by (4.12). The geodesic curvatures η_L and η_M of the initial fibre trajectories are obtained simply by setting $\lambda = 1$, $\mu = 1$ and $\gamma = 0$, identically:

$$\eta_L = -M^{\alpha}|_{\alpha}, \quad \eta_M = L^{\alpha}|_{\alpha}. \tag{5.22}$$

The Tchebychev curvatures can be expressed in terms of λ, μ, γ and the initial fibre vectors as follows: We impose the condition that the basis vectors $\boldsymbol{a}_{\alpha}$ in (4.13) be holonomic, as in (2.1). Thus if these vectors are continuously differentiable functions of θ^1 and θ^2, we require that

$$\mu^{\alpha\beta}(\lambda \boldsymbol{l} L_{\alpha} + \mu \boldsymbol{m} M_{\alpha})_{,\beta} = \mathbf{0}. \tag{5.23}$$

From (2.20*b*) and formulae analogous to (5.17) and (5.18) for η_L and η_M, we obtain

$$M^{\alpha}(\lambda \boldsymbol{l})_{,\alpha} - L^{\alpha}(\mu \boldsymbol{m})_{,\alpha} = \lambda \boldsymbol{l} \eta_L + \mu \boldsymbol{m} \eta_M. \tag{5.24}$$

Then (4.6) and (5.14*b*, *c*) result in

$$\lambda \boldsymbol{l} \eta_L + \mu \boldsymbol{m} \eta_M = (\boldsymbol{M} \cdot \nabla \lambda)\, \boldsymbol{l} - (\boldsymbol{L} \cdot \nabla \mu)\, \boldsymbol{m} + \lambda \mu (\phi_l \boldsymbol{p} - \phi_m \boldsymbol{q}). \tag{5.25}$$

We scalar-multiply by $\boldsymbol{m}$ and note from (4.14) that

$$\boldsymbol{p} \cdot \boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{l} \cdot \boldsymbol{m} = \boldsymbol{n} \cdot \boldsymbol{l} \times \boldsymbol{m} = |\cos \gamma|. \tag{5.26}$$

This, together with (4.8), leads to the result

$$J \phi_l = J \eta_m + (\lambda \cos \gamma)\, \boldsymbol{M} \cdot \nabla \gamma. \tag{5.27}$$

A similar procedure provides

$$J \phi_m = J \eta_l - (\mu \cos \gamma)\, \boldsymbol{L} \cdot \nabla \gamma. \tag{5.28}$$

From these formulae it is evident that for the special case of an orthogonal network ($\gamma = 0$), the Tchebychev curvature of a family of fibres is equal to the geodesic curvature of the orthogonal family. In particular, this conclusion applies to the fibre trajectories on the reference surface.

(*c*) *The equations of Gauss and Mainardi–Codazzi*

The various geometric variables that have been introduced in this section are not independent. They must be restricted so as to ensure that equations (5.12) are integrable, i.e.

$$\mu^{\beta\gamma}(x_{\beta} \boldsymbol{p} + u_{\beta} \boldsymbol{n})_{,\gamma} = \mathbf{0}, \quad \mu^{\beta\gamma}(y_{\beta} \boldsymbol{q} + v_{\beta} \boldsymbol{n})_{,\gamma} = \mathbf{0}. \tag{5.29}$$

From these we obtain the well-known Gauss and Mainardi–Codazzi compatibility equations in forms that are useful in the applications considered here. According to (2.10) and (5.11*a*) and (5.12),

$$\boldsymbol{n}_{,\gamma} = -b_{\alpha\gamma} \boldsymbol{a}^{\alpha} \quad \text{and} \quad \boldsymbol{p}_{,\gamma} = -b_{\alpha\gamma} \boldsymbol{a}^{\alpha} \times \boldsymbol{l} - x_{\gamma} \boldsymbol{l}. \tag{5.30}$$

Substituting these into (5.29*a*) and making use of (5.16) and (5.13*c*, *d*), we obtain

$$\mu^{\beta\gamma}[(x_{\beta,\gamma} - u_{\beta} b_{\alpha\gamma} p^{\alpha})\, \boldsymbol{p} + (x_{\beta} b_{\alpha\gamma} p^{\alpha} + u_{\beta,\gamma})\, \boldsymbol{n}] = \mathbf{0}, \tag{5.31}$$

from which it follows that

$$\mu^{\beta\gamma}(x_{\beta,\gamma} - u_{\beta} b_{\alpha\gamma} p^{\alpha}) = 0 \quad \text{and} \quad \mu^{\beta\gamma}(x_{\beta} b_{\alpha\gamma} p^{\alpha} + u_{\beta,\gamma}) = 0. \tag{5.32}$$

A similar procedure applied to (5.29*b*) furnishes the system

$$\mu^{\beta\gamma}(y_{\beta,\gamma} - v_{\beta} b_{\alpha\gamma} q^{\alpha}) = 0 \quad \text{and} \quad \mu^{\beta\gamma}(y_{\beta} b_{\alpha\gamma} q^{\alpha} + v_{\beta,\gamma}) = 0. \tag{5.33}$$

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After much effort we find that (5.32a) and (5.33a) both reduce to the Gauss equation (see Appendix):

$$J\kappa = (\lambda\eta_l M^\alpha - \mu\eta_m L^\alpha)|_\alpha - \tfrac{1}{2}J^{-1}\lambda\mu\cos\gamma[(L^\alpha M^\beta + M^\alpha L^\beta)\gamma_{,\alpha}]|_\beta. \tag{5.34}$$

Here κ is the gaussian curvature, given by (5.8). From (5.32b) and (5.33b) we derive the Mainardi–Codazzi equations

$$\left.\begin{aligned} J(\lambda\mu)^{-2}(\mu\tau L^\alpha - \lambda\kappa_l M^\alpha)|_\alpha &= \eta_l(\kappa_m - \tau\sin\gamma) + \phi_l(\kappa_l\sin\gamma - \tau),\\ J(\lambda\mu)^{-2}(\mu\kappa_m L^\alpha - \lambda\tau M^\alpha)|_\alpha &= \phi_m(\kappa_m\sin\gamma - \tau) + \eta_m(\kappa_l - \tau\sin\gamma),\end{aligned}\right\} \tag{5.35}$$

respectively. These latter equations are also the integrability conditions for (5.30a).

Let κ_L and κ_M be the normal curvatures of the $\boldsymbol{L}$- and $\boldsymbol{M}$-trajectories, respectively, and let $\bar{\tau}$ be the torsion. Then we can read off the Gauss and Mainardi–Codazzi equations for the reference surface from (5.34) and (5.35):

$$\left.\begin{aligned} \bar{\kappa} &= (\eta_L M^\alpha - \eta_M L^\alpha)|_\alpha,\\ (\bar{\tau}L^\alpha - \kappa_L M^\alpha)|_\alpha &= \eta_L\kappa_M - \eta_M\bar{\tau},\\ (\kappa_M L^\alpha - \bar{\tau}M^\alpha)|_\alpha &= \eta_M\kappa_L - \eta_L\bar{\tau}.\end{aligned}\right\} \tag{5.36}$$

Here

$$\bar{\kappa} = \kappa_L\kappa_M - \bar{\tau}^2 \tag{5.37}$$

is the gaussian curvature of the reference surface. The mean curvature is given by

$$\bar{H} = \tfrac{1}{2}(\kappa_L + \kappa_M). \tag{5.38}$$

6. Energy minimizers, convexity and rank-one convexity

In this section we characterize those deformations $\boldsymbol{r}(\theta^1, \theta^2)$ that minimize the potential energy $E[\cdot]$ locally, in the sense that

$$E[\boldsymbol{r}] \leqslant E[\boldsymbol{r} + \Delta\boldsymbol{r}], \tag{6.1}$$

for all sufficiently small perturbations $\Delta\boldsymbol{r}(\theta^1, \theta^2)$ that vanish on ∂S_r:

$$|\Delta\boldsymbol{r}| < \epsilon, \quad (\theta^1, \theta^2) \in S; \quad \Delta\boldsymbol{r} = \boldsymbol{0}, \quad (\theta^1, \theta^2) \in \partial S_r. \tag{6.2}$$

We assume that $\boldsymbol{r}(\cdot)$ is continuously differentiable.

Graves (1939) has shown that if $\boldsymbol{r}$ is a local minimizer, then $W(\cdot)$ is rank-one convex at $\boldsymbol{F} = \boldsymbol{r}_{,\alpha} \otimes \boldsymbol{A}^\alpha$:

$$W(\boldsymbol{F} + \boldsymbol{a}\otimes\boldsymbol{b}) - W(\boldsymbol{F}) \geqslant \boldsymbol{a}\cdot\boldsymbol{T}(\boldsymbol{F})\,\boldsymbol{b}, \tag{6.3}$$

for all $\boldsymbol{a}$ and $\boldsymbol{b}$ of the form

$$\boldsymbol{a} = a^\alpha\boldsymbol{a}_\alpha + a\boldsymbol{n}, \quad \boldsymbol{b} = b_\alpha\boldsymbol{A}^\alpha, \tag{6.4}$$

and every $(\theta^1, \theta^2) \in S$. $W(\cdot)$ is said to be rank-one convex (without qualification) if (6.3) is satisfied for all $\boldsymbol{F}$. Graves proved (6.3) for functionals of the form (3.9) with $P[\boldsymbol{r}]$ absent. The result is known to be valid for the problem of dead-loading (Truesdell & Noll 1965, §68 bis). For the problem of pressure loading the necessity of (6.3) follows from the property that $\chi(\boldsymbol{r}; \boldsymbol{r}_{,\alpha})$ is rank-one affine (Steigmann 1986, 1991):

$$\chi(\boldsymbol{r}; \boldsymbol{r}_{,\alpha} + \boldsymbol{a}b_\alpha) - \chi(\boldsymbol{r}; \boldsymbol{r}_{,\alpha}) \equiv a^i b_\alpha(\partial\chi/\partial r^i_{,\alpha}); \quad a^i = \boldsymbol{a}\cdot\boldsymbol{e}_i. \tag{6.5}$$

For elastic nets we show that $W(\cdot)$ is rank-one convex at $\boldsymbol{F}$ if and only if it is convex at $\boldsymbol{F}$, i.e.

$$W(\boldsymbol{F}+\Delta\boldsymbol{F})-W(\boldsymbol{F}) \geqslant \boldsymbol{T}(\boldsymbol{F}):\Delta\boldsymbol{F}, \quad \forall\Delta\boldsymbol{F}. \tag{6.6}$$

The notation $\boldsymbol{A}:\boldsymbol{B}$ is used to denote the scalar product, $\operatorname{tr}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B})$, of tensors $\boldsymbol{A}$ and $\boldsymbol{B}$. $W(\cdot)$ is said to be convex if (6.6) is satisfied for all $\boldsymbol{F}$. Sufficiency follows simply by setting $\Delta\boldsymbol{F} = \boldsymbol{a}\otimes\boldsymbol{b}$ in (6.6). To prove necessity, we first show that $W(\cdot)$ is rank-one convex at $\boldsymbol{F}$ only if $F(\cdot)$ and $G(\cdot)$ are convex and non-decreasing at $\lambda = |\boldsymbol{FL}|$ and $\mu = |\boldsymbol{FM}|$, respectively, i.e.

$$F(\lambda+\alpha)-F(\lambda) \geqslant \alpha f(\lambda), \quad G(\mu+\beta)-G(\mu) \geqslant \beta g(\mu) \tag{6.7}$$

and

$$f(\lambda) \geqslant 0, \quad g(\mu) \geqslant 0 \tag{6.8}$$

for all α and β such that $\lambda+\alpha \geqslant 0$ and $\mu+\beta \geqslant 0$. We then show that if the fibre stretches λ and μ delivered by $\boldsymbol{F}$ satisfy (6.7, 8), then $\boldsymbol{F}$ satisfies (6.6).

To prove these statements we begin by recording some auxiliary kinematical identities. Let

$$\boldsymbol{F}^* = \boldsymbol{F}+\Delta\boldsymbol{F}, \quad \boldsymbol{C}^* = (\boldsymbol{F}^*)^{\mathrm{T}}\boldsymbol{F}^*. \tag{6.9}$$

Then

$$\boldsymbol{C}^* = \boldsymbol{C}+\boldsymbol{F}^{\mathrm{T}}(\Delta\boldsymbol{F})+(\Delta\boldsymbol{F})^{\mathrm{T}}\boldsymbol{F}+(\Delta\boldsymbol{F})^{\mathrm{T}}(\Delta\boldsymbol{F}), \tag{6.10}$$

where $\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$. The fibre stretches associated with $\boldsymbol{F}^*$ are given by $\lambda^* = (\boldsymbol{L}\cdot\boldsymbol{C}^*\boldsymbol{L})^{\frac{1}{2}}$ and $\mu^* = (\boldsymbol{M}\cdot\boldsymbol{C}^*\boldsymbol{M})^{\frac{1}{2}}$. From (4.1), (4.2) and (6.10) we obtain λ^* and μ^* as the non-negative roots of

$$\left.\begin{aligned}(\lambda^*)^2 &= \lambda^2+2\lambda\boldsymbol{l}\cdot(\Delta\boldsymbol{F})\,\boldsymbol{L}+|(\Delta\boldsymbol{F})\,\boldsymbol{L}|^2,\\ (\mu^*)^2 &= \mu^2+2\mu\boldsymbol{m}\cdot(\Delta\boldsymbol{F})\,\boldsymbol{M}+|(\Delta\boldsymbol{F})\,\boldsymbol{M}|^2.\end{aligned}\right\} \tag{6.11}$$

For $\Delta\boldsymbol{F} = \boldsymbol{a}\otimes\boldsymbol{b}$ these become

$$\left.\begin{aligned}(\lambda^*)^2 &= \lambda^2+2\lambda(\boldsymbol{a}\cdot\boldsymbol{l})(\boldsymbol{b}\cdot\boldsymbol{L})+|\boldsymbol{a}|^2(\boldsymbol{b}\cdot\boldsymbol{L})^2,\\ (\mu^*)^2 &= \mu^2+2\mu(\boldsymbol{a}\cdot\boldsymbol{m})(\boldsymbol{b}\cdot\boldsymbol{M})+|\boldsymbol{a}|^2(\boldsymbol{b}\cdot\boldsymbol{M})^2.\end{aligned}\right\} \tag{6.12}$$

According to (4.22),

$$\boldsymbol{a}\cdot\boldsymbol{T}(\boldsymbol{F})\,\boldsymbol{b} = f(\lambda)\,(\boldsymbol{a}\cdot\boldsymbol{l})\,(\boldsymbol{b}\cdot\boldsymbol{L})+g(\mu)\,(\boldsymbol{a}\cdot\boldsymbol{m})\,(\boldsymbol{b}\cdot\boldsymbol{M}). \tag{6.13}$$

Then for elastic nets (6.3) is equivalent to the inequality

$$F(\lambda^*)+G(\mu^*)-F(\lambda)-G(\mu) \geqslant f(\lambda)\,(\boldsymbol{a}\cdot\boldsymbol{l})\,(\boldsymbol{b}\cdot\boldsymbol{L})+g(\mu)\,(\boldsymbol{a}\cdot\boldsymbol{m})\,(\boldsymbol{b}\cdot\boldsymbol{M}), \tag{6.14}$$

where λ^* and μ^* are the non-negative roots of (6.12). To examine the implications of (6.14), set $\boldsymbol{a} = a\boldsymbol{n}$. Then (4.14) and (6.12, 14) imply that

$$F(\lambda^*)+G(\mu^*) \geqslant F(\lambda)+G(\mu), \quad \forall\lambda^* \geqslant \lambda \quad \text{and} \quad \mu^* \geqslant \mu. \tag{6.15}$$

Now let $\boldsymbol{b}\cdot\boldsymbol{M} = 0$. Then $\mu^* = \mu$ and we conclude that $F(\lambda^*) \geqslant F(\lambda)$, $\forall\lambda^* \geqslant \lambda$. Similarly, by setting $\boldsymbol{b}\cdot\boldsymbol{L} = 0$ instead we deduce that $G(\mu^*) \geqslant G(\mu), \forall\mu^* \geqslant \mu$. From these results it follows that λ and μ satisfy (6.8), i.e. the fibre stresses delivered by an energy-minimizing deformation are non-negative. Next, let $\boldsymbol{b}\cdot\boldsymbol{M} = 0$ in (6.14) with $\boldsymbol{a}$ arbitrary. Then $\mu^* = \mu$ and (6.14) becomes

$$F(\lambda^*)-F(\lambda) \geqslant f(\lambda)\,(\boldsymbol{a}\cdot\boldsymbol{l})\,(\boldsymbol{b}\cdot\boldsymbol{L}). \tag{6.16}$$

Now choose $\boldsymbol{b} = \boldsymbol{L}$ and $\boldsymbol{a} = \alpha \boldsymbol{l}$ with $\lambda + \alpha \geqslant 0$. Then (6.12*a*) gives $\lambda^* = \lambda + \alpha$ and (6.16) reduces to (6.7*a*). Similarly, by taking $\boldsymbol{b} = \boldsymbol{M}$ and $\boldsymbol{a} = \beta \boldsymbol{m}$ with $\mu + \beta \geqslant 0$, we find that (6.14) reduces to (6.7*b*).

We have shown that $W(\cdot)$ is rank-one convex at $\boldsymbol{F}$ only if $\lambda (= |\boldsymbol{FL}|)$ and $\mu (= |\boldsymbol{FM}|)$ satisfy (6.7) and (6.8). We now show that if λ and μ satisfy (6.7) and (6.8), then $W(\cdot)$ is convex at $\boldsymbol{F}$, i.e. $\boldsymbol{F}$ satisfies (6.6). To this end we combine (6.7*a*, *b*) to get

$$F(\lambda^*) - F(\lambda) + G(\mu^*) - G(\mu) \geqslant (\lambda^* - \lambda) f(\lambda) + (\mu^* - \mu)\, g(\mu). \tag{6.17}$$

From (4.22) it is apparent that (6.6) is equivalent to

$$F(\lambda^*) - F(\lambda) + G(\mu^*) - G(\mu) \geqslant f(\lambda)\, \boldsymbol{l} \cdot (\Delta \boldsymbol{F})\, \boldsymbol{L} + g(\mu)\, \boldsymbol{m} \cdot (\Delta \boldsymbol{F})\, \boldsymbol{M}, \tag{6.18}$$

where
$$\lambda^* = |\boldsymbol{F}^* \boldsymbol{L}|, \quad \mu^* = |\boldsymbol{F}^* \boldsymbol{M}| \quad \text{and} \quad \boldsymbol{F}^* = \boldsymbol{F} + \Delta \boldsymbol{F}. \tag{6.19}$$

From (4.1) and the inequalities $\boldsymbol{l} \cdot \boldsymbol{F}^* \boldsymbol{L} \leqslant |\boldsymbol{F}^* \boldsymbol{L}|$, $\boldsymbol{m} \cdot \boldsymbol{F}^* \boldsymbol{M} \leqslant |\boldsymbol{F}^* \boldsymbol{M}|$ one can easily verify that

$$\lambda^* - \lambda \geqslant \boldsymbol{l} \cdot (\Delta \boldsymbol{F})\, \boldsymbol{L}, \quad \mu^* - \mu \geqslant \boldsymbol{m} \cdot (\Delta \boldsymbol{F})\, \boldsymbol{M}. \tag{6.20}$$

It is then evident that (6.18) follows if (6.8) and (6.17) hold. In summary, we have shown that $W(\cdot)$ is rank-one convex at $\boldsymbol{F}$ if and only if it is convex at $\boldsymbol{F}$. Moreover, these inequalities are equivalent to (6.7) and (6.8), where λ and μ are the fibre stretches delivered by $\boldsymbol{F}$.

Suppose now that $\boldsymbol{r}(\theta^1, \theta^2)$ is an equilibrium deformation for the problem of dead loading (§3). Then $\boldsymbol{r}$ is a solution of (3.13) and (3.14) with $p = 0$. Let $\boldsymbol{F}$ be the gradient of $\boldsymbol{r}$ and suppose the associated fibre stretches satisfy (6.7) and (6.8). From (3.9), (3.12), (3.14), (6.1) and (6.2*b*),

$$E[\boldsymbol{r} + \Delta \boldsymbol{r}] - E[\boldsymbol{r}] = \iint_S [W(\boldsymbol{F} + \Delta \boldsymbol{F}) - W(\boldsymbol{F})]\, A^{\frac{1}{2}}\, \mathrm{d}\theta^1\, \mathrm{d}\theta^2 - \oint_{\boldsymbol{x}(\partial S)} \boldsymbol{T}(\boldsymbol{F})\, \boldsymbol{\nu} \cdot \Delta \boldsymbol{r}\, \mathrm{d}s, \tag{6.21}$$

where $\Delta \boldsymbol{F} = (\Delta \boldsymbol{r})_{,\alpha} \otimes \boldsymbol{A}^{\alpha}$. Because $\boldsymbol{r}$ is an equilibrium deformation, we can use the Green–Stokes theorem to obtain

$$E[\boldsymbol{r} + \Delta \boldsymbol{r}] = E[\boldsymbol{r}] + \iint_S [W(\boldsymbol{F} + \Delta \boldsymbol{F}) - W(\boldsymbol{F}) - \boldsymbol{T}(\boldsymbol{F}) : \Delta \boldsymbol{F}]\, A^{\frac{1}{2}}\, \mathrm{d}\theta^1\, \mathrm{d}\theta^2. \tag{6.22}$$

The integral is non-negative because $\boldsymbol{F}$ satisfies (6.6). Furthermore, since $|\Delta \boldsymbol{r}|$ was not restricted in the development leading to (6.18) (or (6.6)), it follows that $E[\boldsymbol{r}]$ is the absolute minimum energy. Thus for the problem of dead loading of an elastic net, all local minimizers are global minimizers as well.

7. Relaxed energy density and continuously distributed wrinkles

In addition to the hypotheses on the fibre response functions F, G, f, g stated in §4, we assume that f and g are positive (respectively negative) if λ and μ are greater (respectively less) than unity, together with

$$F, G \to \infty \quad \text{as} \quad \lambda, \mu \to 0. \tag{7.1}$$

Our assumptions imply that inequalities (6.8) are violated when λ or $\mu \in [0, 1)$. Thus deformations that deliver fibre stretches in this interval cannot be energy minimizers. One might therefore attempt to find equilibrium deformations for which $\lambda \geqslant 1$ and

$\mu \geqslant 1$ at all points of the material domain S. For example, deformations that furnish stretches $\mu = 1$ and $\lambda > 1$ correspond to uniaxial states of stress and satisfy (6.8). It is easy to see that solutions of this kind do not exist, however: the equilibrium and compatibility equations (3.13), (5.34) and (5.35) constitute six restrictions on the six variables λ, μ, γ, κ_l, κ_m and τ. Further restrictions on λ and μ would lead to an overdetermined system having no solution except in certain trivial cases. There may also exist intervals on which (6.7) fail. We conclude that an equilibrium deformation may violate (6.3) on some parts of the material domain so that $E[\cdot]$ will generally fail to have a minimizer.

(*a*) *Relaxed energy density*

To overcome this difficulty we use a certain relaxed energy density W_q, constructed in such a way that it always satisfies (6.7) and (6.8). W_q is defined to be the quasi-convexification of W (Dacorogna 1982, 1989):

$$W_q(\boldsymbol{F}) = \sup_{\phi}\{\phi(\boldsymbol{F})A^{\frac{1}{2}} \quad \text{quasi-convex and} \quad \phi \leqslant W\}. \tag{7.2}$$

A function $U(\boldsymbol{r}_0, \cdot\,; \xi^1, \xi^2)$ is quasi-convex at $\boldsymbol{F}$ if and only if

$$\iint_D U(\boldsymbol{r}_0, \boldsymbol{F}+\Delta\boldsymbol{F}; \xi^1, \xi^2)\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2 \geqslant U(\boldsymbol{r}_0, \boldsymbol{F}; \xi^1, \xi^2)\iint_D \mathrm{d}\theta^1\,\mathrm{d}\theta^2 \tag{7.3}$$

for each fixed deformation $\boldsymbol{r}_0$, each fixed point $(\xi^1, \xi^2) \in S$, for all $D \subset S$ and for every $\Delta\boldsymbol{F} = \boldsymbol{u}_{,\alpha} \otimes \boldsymbol{A}^{\alpha}$ with $\boldsymbol{u} = \boldsymbol{0}$ on ∂D. $U(\boldsymbol{r}_0, \cdot\,; \xi^1, \xi^2)$ is quasi-convex (without qualification) if (7.3) is valid for all $\boldsymbol{F}$.

From the Green–Stokes theorem and Graves's theorem it is easily demonstrated that $U(\boldsymbol{r}_0, \cdot\,; \xi^1, \xi^2)$ is quasi-convex if it is convex and only if it is rank-one convex (Ball 1977*a*, *b*). We define the rank-one convexification W_r of W and the convexification W_c of W by

$$W_r(\boldsymbol{F}) = \sup_{\phi}\{\phi(\boldsymbol{F}) \quad \text{rank-one convex and} \quad \phi \leqslant W\}, \tag{7.4}$$

$$W_c(\boldsymbol{F}) = \sup_{\phi}\{\phi(\boldsymbol{F}) \quad \text{convex and} \quad \phi \leqslant W\}, \tag{7.5}$$

respectively. Then since W_c, W_q and W_r belong to successively larger classes of functions, it follows that

$$W_c \leqslant W_q \leqslant W_r. \tag{7.6}$$

Although no general algorithm for the computation of W_q is known, it may happen that for a particular W the functions W_c and W_r can be computed explicitly. If it is found that $W_c = W_r$ then the bounds (7.6) furnish W_q directly (Kohn & Strang 1986; Pipkin 1986*b*, 1989). This is precisely the case for $W(\boldsymbol{F}) = \hat{W}(\boldsymbol{C})$ defined by (4.17). For, the analysis of §6 demonstrates that $W(\cdot)$ is convex if and only if it is rank-one convex. It follows that $W_c = W_r$. Furthermore, our analysis indicates that convexity and rank-one convexity of $W(\boldsymbol{F})$ are equivalent to the inequalities (6.7) and (6.8), where $\lambda = |\boldsymbol{FL}|$ and $\mu = |\boldsymbol{FM}|$. From these results we immediately deduce that

$$W_q(\boldsymbol{F}) = F_c(\lambda) + G_c(\mu), \tag{7.7}$$

where

$$\{F_c(\cdot),\, G_c(\cdot)\} = \sup_{\phi}\{\phi(\cdot) \quad \text{convex, non-decreasing and} \quad \phi \leqslant F, G\}. \tag{7.8}$$

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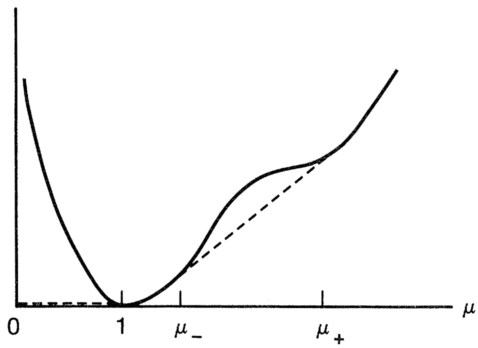

Figure 1. Relaxed fibre strain energy. ——, $G(\cdot)$; - - - -, $G_c(\cdot)$.

Thus F_c and G_c vanish in the interval $[0,1]$ and are given by the lower convex envelopes of F and G, respectively, when $(\lambda,\mu) > 1$. A sketch of G_c for a particular G is given in figure 1. F_c is constructed from F similarly.

According to a relaxation theorem of Dacorogna (1989, §5.2.1), minimizers of the functional $E_q[\cdot]$, obtained by replacing W by W_q in (3.9) and (3.12) with $\partial S = \partial S_r$, can be regarded as minimizers of the corresponding problem for $E[\cdot]$ in the following sense: If $\boldsymbol{r}$ minimizes E_q then there is a minimizing sequence $\boldsymbol{r}_n$ for E, converging weakly to $\boldsymbol{r}$ in S, for which $E[\boldsymbol{r}_n] \rightarrow E_q[\boldsymbol{r}]$. Under suitable supplementary conditions on W_q, the existence of a minimizer for E_q is assured (Morrey 1952). We refer to Dacorogna (1989) for precise statements of these and related results.

(b) *Continuously distributed wrinkles*

Unfortunately Dacorogna's theorem is not directly applicable here because its proof requires that W satisfy growth conditions that are not consistent with (7.1). Thus to motivate the use of W_q for λ or $\mu \in [0,1]$, we will appeal to a formal construction involving the notion of a continuous distribution of wrinkles (Pipkin 1986*b*). This construction furnishes a strain energy that coincides with W_q when λ or μ belong to the interval $[0,1]$. The quasi-convexification of this strain energy for all $\lambda,\mu \geqslant 0$ is then given by (7.7) and (7.8).

Thus consider a sequence $\boldsymbol{r}_j; j = 1,2,\ldots$ of deformations with discontinuous gradients $\boldsymbol{F}_j$. Suppose the fibre stretches λ,μ associated with $\boldsymbol{F}_j$ are independent of j, with $\lambda > 1$ and $\mu = 1$:

$$(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F})_j = \boldsymbol{C}_j = \lambda^2 \boldsymbol{L}\otimes\boldsymbol{L} + \boldsymbol{M}\otimes\boldsymbol{M} + \lambda\sin\gamma_j(\boldsymbol{L}\otimes\boldsymbol{M} + \boldsymbol{M}\otimes\boldsymbol{L}). \tag{7.9}$$

Then the strain energy W has the same value for every member of the sequence:

$$W(\boldsymbol{F}_j) = F(\lambda) + G(1) = F(\lambda); \quad j = 1,2,\ldots. \tag{7.10}$$

Let functions $\xi(\theta^1,\theta^2)$ and $\eta(\theta^1,\theta^2)$ parametrize the curves with unit tangents $\boldsymbol{L}$ and $\boldsymbol{M}$, respectively, i.e.

$$\partial \boldsymbol{x}/\partial\xi = \alpha\boldsymbol{L}, \quad \partial\boldsymbol{x}/\partial\eta = \beta\boldsymbol{M}, \tag{7.11}$$

where $\alpha = |\partial\boldsymbol{x}/\partial\xi|$ and $\beta = |\partial\boldsymbol{x}/\partial\eta|$ are assumed to be strictly positive. For the jth member of the sequence, let the membrane be folded along each of the K_j curves $\theta^\alpha = f_j^\alpha(\xi,\eta_{jk})$; $k = 1,\ldots,K_j$. Let $K_j \rightarrow \infty$ as $j \rightarrow \infty$ while the spacings between folds approach zero, in such a way that the sequences $f_j^\alpha(\xi,\eta)$ have smooth limits $f^\alpha(\xi,\eta)$. We regard the curve $\theta^\alpha = f^\alpha(\cdot,\eta)$ as a fold for each η. Thus the limiting deformation contains a continuous distribution of wrinkles.

Each member $\boldsymbol{r}_j$ of the sequence partitions the membrane into strips between the K_j curves. We label these strips $(+)$ and $(-)$ alternately. Let $I_j(\eta)$ be the indicator function for $(+)$ strips in the jth partition:

$$I_j(\eta) = \begin{cases} 1, & \eta \in (+), \\ 0, & \eta \in (-). \end{cases} \tag{7.12}$$

We require that the sequence of partitions be such that

$$\lim \int_{\eta_0}^{\eta} I_j(x)\,\mathrm{d}x = \int_{\eta_0}^{\eta} \theta(x)\,\mathrm{d}x \tag{7.13}$$

for all η_0 and η. Then $\theta(\eta)$ is the fractional density of $(+)$ strips in the limiting deformation and $0 \leqslant \theta(\eta) \leqslant 1$. Consider a sequence $h_j(\eta)$ of continuous functions and suppose $h_j \rightarrow h$ uniformly. Then

$$\lim \int_{\eta_0}^{\eta} h_j(x)\, I_j(x)\,\mathrm{d}x = \int_{\eta_0}^{\eta} h(x)\,\theta(x)\,\mathrm{d}x. \tag{7.14}$$

From (7.11) we find the directional derivatives $\boldsymbol{L}\cdot\nabla = \alpha^{-1}\partial/\partial\xi$ and $\boldsymbol{M}\cdot\nabla = \beta^{-1}\partial/\partial\eta$. Then equations (4.4) give

$$\partial \boldsymbol{r}_j/\partial\xi = \alpha\lambda \boldsymbol{l}_j, \quad \partial \boldsymbol{r}_j/\partial\eta = \beta \boldsymbol{m}_j, \tag{7.15}$$

where $\boldsymbol{l}_j$ and $\boldsymbol{m}_j$ are the deformed fibre trajectories. We assume that the $\boldsymbol{l}_j$ are continuous and

$$\boldsymbol{m}_j = \left\{\begin{matrix} \boldsymbol{u}_j^+, & \eta \in (+), \\ \boldsymbol{u}_j^-, & \eta \in (-), \end{matrix}\right\} \quad |\boldsymbol{u}_j^{\pm}| = 1, \tag{7.16}$$

for some continuous functions $\boldsymbol{u}_j^{\pm}$. We find that discontinuities in $\boldsymbol{F}_j$ at the folds are of the form

$$\boldsymbol{F}_j^+ - \boldsymbol{F}_j^- = \boldsymbol{a}_j \otimes \boldsymbol{M}, \tag{7.17}$$

where $\boldsymbol{a}_j = \boldsymbol{u}_j^+ - \boldsymbol{u}_j^-$. Thus the derivatives (7.15) are geometrically admissible (Truesdell & Toupin 1960, ch. C). Of course, for specified functions λ, α and β, the $\boldsymbol{l}_j$, $\boldsymbol{u}_j^{\pm}$ and their derivatives must be restricted in such a way as to ensure the existence of $\mathbf{r}_j$ between folds.

The jth deformation is given by the path-independent integral:

$$\boldsymbol{r}_j = \boldsymbol{r}_0 + \int_{\xi_0}^{\xi} \lambda \boldsymbol{l}_j \alpha\,\mathrm{d}\xi' + \int_{\eta_0}^{\eta} [I_j \boldsymbol{u}_j^+ + (1-I_j)\,\boldsymbol{u}_j^-]\,\beta\,\mathrm{d}\eta', \tag{7.18}$$

where $\boldsymbol{r}_0 = \boldsymbol{r}(\xi_0, \eta_0)$ is fixed for the sake of convenience and the integrals are evaluated on arcs of piecewise constant η and ξ. We now assume that $\boldsymbol{l}_j \rightarrow \boldsymbol{l}$ and $\boldsymbol{u}_j^{\pm} \rightarrow \boldsymbol{u}_{\pm}$, where $\boldsymbol{l}$ and $\boldsymbol{u}_{\pm}$ are continuous. Then $\boldsymbol{r}_j \rightarrow \boldsymbol{r}$, where

$$\boldsymbol{r} = \boldsymbol{r}_0 + \int_{\xi_0}^{\xi} \lambda \boldsymbol{l} \alpha\,\mathrm{d}\xi' + \int_{\eta_0}^{\eta} [\theta \boldsymbol{u}_+ + (1-\theta)\,\boldsymbol{u}_-]\,\beta\,\mathrm{d}\eta'. \tag{7.19}$$

The gradient $\boldsymbol{F}$ of the limiting deformation is continuous:

$$\boldsymbol{F} = \lambda \boldsymbol{l} \otimes \boldsymbol{L} + \mu \boldsymbol{m} \otimes \boldsymbol{M}, \tag{7.20}$$

where
$$\mu \boldsymbol{m} = \beta^{-1}\partial \boldsymbol{r}/\partial\eta = \theta \boldsymbol{u}_+ + (1-\theta)\,\boldsymbol{u}_-. \tag{7.21}$$

We can take $|\boldsymbol{m}| = 1$ without loss of generality. Since $|\boldsymbol{u}_{\pm}| = 1$ it follows that μ can take on any value in the interval $[0,1]$. According to (7.10) the strain energy associated with the limiting deformation (7.19) is $F(\lambda)$ and is thus independent of μ.

Thus we find that continuously differentiable deformations with fibre stretches $\lambda > 1$ and $0 \leqslant \mu \leqslant 1$ are obtained as limits of sequences of finely wrinkled configurations without compressive stress. The associated strain energy has the value $F(\lambda)+G(1)$, where $G(1) = 0$ is the minimum value of $G(\cdot)$. Similarly, deformations for which $0 \leqslant (\lambda, \mu) \leqslant 1$ can be achieved by double wrinkling (Pipkin 1986*b*) with no stress at all. The strain energy attributed to such a deformation has the value $F(1)+G(1) = 0$.

(*c*) *Fine scale phase mixtures*

If $G(\cdot)$ is non-convex in some interval $\mu_- \leqslant \mu \leqslant \mu_+$ (figure 1), then a similar construction leads to an interpretation of $G_c(\cdot)$ in terms of a fine distribution of phases interspersed in strips in which $\mu = \mu_+$ and μ_- alternately. For example, consider a sequence $\boldsymbol{F}_j$ with strain

$$\boldsymbol{C}_j = \lambda^2 \boldsymbol{L}\otimes\boldsymbol{L} + \mu_j^2 \boldsymbol{M}\otimes\boldsymbol{M} + \lambda\mu_j \sin\gamma_j(\boldsymbol{L}\otimes\boldsymbol{M}+\boldsymbol{M}\otimes\boldsymbol{L}), \tag{7.22}$$

where

$$\mu_j = \begin{cases} \mu_+, & \eta\in(+), \\ \mu_-, & \eta\in(-). \end{cases} \tag{7.23}$$

Take the $\boldsymbol{l}_j$ and $\boldsymbol{m}_j$ to be continuous with continuous limits $\boldsymbol{l}$ and $\boldsymbol{m}$, respectively. Then we find $\boldsymbol{r}_j \to \boldsymbol{r}$, with gradient

$$\boldsymbol{F} = \lambda\boldsymbol{l}\otimes\boldsymbol{L} + \mu\boldsymbol{m}\otimes\boldsymbol{M}, \tag{7.24}$$

where $\mu = \theta\mu_+ + (1-\theta)\mu_-$ has values in $[\mu_-, \mu_+]$. From (7.23), the fibre stress g associated with $\boldsymbol{r}_j$ is the same for all j:

$$g(\mu_j) = g(\mu_{\pm}) = k, \tag{7.25}$$

where k is the Maxwell stress, defined by

$$G(\mu_+) - G(\mu_-) = k(\mu_+ - \mu_-). \tag{7.26}$$

(*d*) *Pressure loading*

Finally, we consider the potential energy

$$E[\boldsymbol{r}] = \iint_S [W(\boldsymbol{F})A^{\frac{1}{2}} - p\chi(\boldsymbol{r};\boldsymbol{r}_{,\alpha})]\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2 \tag{7.27}$$

of a net inflated by constant pressure p (§3). For fixed $\boldsymbol{r}_0$, we define

$$\psi(\cdot) = \chi(\boldsymbol{r}_0;\cdot). \tag{7.28}$$

Then from (3.11) we compute

$$(\partial\psi/\partial r^i_{,\alpha})_{,\alpha}\,\boldsymbol{e}_i = \tfrac{1}{3}e^{\alpha\beta}\boldsymbol{r}_{,\alpha\beta}\times\boldsymbol{r}_0, \tag{7.29}$$

which vanishes if $\boldsymbol{r}$ is twice continuously differentiable. Thus ψ is a null-lagrangian. It follows that $\chi(\boldsymbol{r}_0;\cdot)$ is quasi-affine, satisfying (7.3) as an equality (Ball 1976, 1977*a*). Thus the quasi-convexification of the integrand in (7.27) is obtained by replacing W by W_{q} as before.

We are not aware of an existence theory or relaxation theorems for functionals of the form (3.9)–(3.11) involving volume-dependent pressures. It is clear that any

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results for such functionals will require restrictions on $W(\cdot)$ and $p(\cdot)$ jointly, for otherwise examples can be constructed for which E is not bounded below. Lacking an appropriate theory, we shall assume that the relaxed problem consists of (3.9)–(3.11) with W_{q} replacing W, for then, and only then, the necessary condition of rank-one convexity is satisfied for all $\boldsymbol{F}$. Henceforth we shall confine our attention to the relaxed problem and drop subscripts q and c in (7.7) and (7.8).

8. Equilibrium

In the remainder of this work we use the relaxed energy density to study the equilibrium configurations of elastic nets. The fibre stresses $f(\lambda)$ and $g(\mu)$ associated with this energy are simply the derivatives of the relaxed fibre strain energies defined by (7.8). In particular, $f(\cdot)$ and $g(\cdot)$ vanish if their arguments belong to the interval $[0, 1]$ and are strictly positive and non-decreasing if their arguments exceed unity. This observation leads us naturally to address the equilibrium problem according to the values of the fibre stretches. For example, if λ and $\mu \in [0, 1]$ then both fibre stresses vanish. In this case the net is said to be slack. Such a configuration cannot be maintained in the presence of a lateral pressure. However, if the pressure vanishes then a slack deformation is automatically equilibrated and is therefore highly non-unique. We shall not consider slack nets further.

If one of the stretches belongs to the interval $[0, 1]$ and the other exceeds unity, then there is only one active family of fibres and the net is said to be half-slack. These configurations are associated with the continuously distributed wrinkles of the previous section. The equilibrium theory of half-slack nets is considered in detail in §11. Finally, if both fibre stretches exceed unity then the fibre stresses are positive and the net is said to be tense. The equilibrium equations for a tense net are obtained simply by substituting (4.21) into (3.13). First we note that

$$A^{-\frac{1}{2}}(A^{\frac{1}{2}}\boldsymbol{T}^{\alpha})_{,\alpha} = (fL^{\alpha})|_{\alpha}\,\boldsymbol{l} + (gM^{\alpha})|_{\alpha}\,\boldsymbol{m} + fL^{\alpha}\boldsymbol{l}_{,\alpha} + gM^{\alpha}\boldsymbol{m}_{,\alpha}. \tag{8.1}$$

Then with the aid of (4.6) and (5.14a, b) we can write (3.13) in the form

$$(fL^{\alpha})|_{\alpha}\,\boldsymbol{l} + (gM^{\alpha})|_{\alpha}\,\boldsymbol{m} + \lambda f\eta_l\,\boldsymbol{p} + \mu g\eta_m\,\boldsymbol{q} + (\lambda f\kappa_l + \mu g\kappa_m + pJ)\,\boldsymbol{n} = \boldsymbol{0}. \tag{8.2}$$

Successive scalar-multiplication by $\boldsymbol{n}$, $\boldsymbol{p}$ and $\boldsymbol{q}$, together with (4.8) and (5.11), leads to the system

$$\left.\begin{aligned} \lambda f\kappa_l + \mu g\kappa_m + pJ &= 0,\\ |\cos\gamma|(gM^{\alpha})|_{\alpha} + \lambda f\eta_l + \mu g\eta_m \sin\gamma &= 0,\\ -|\cos\gamma|(fL^{\alpha})|_{\alpha} + \lambda f\eta_l \sin\gamma + \mu g\eta_m &= 0. \end{aligned}\right\} \tag{8.3}$$

The equilibrium equations for a half-slack net are obtained by specializing (8.2). For example, if $\mu \in [0, 1]$ then $g \equiv 0$ and

$$(fL^{\alpha})|_{\alpha}\,\boldsymbol{l} + \lambda f(\eta_l\,\boldsymbol{p} + \kappa_l\,\boldsymbol{n}) + pJ\boldsymbol{n} = \boldsymbol{0}. \tag{8.4}$$

Then it follows that

$$(fL^{\alpha})|_{\alpha} = 0, \quad \eta_l = 0, \quad \lambda f\kappa_l + pJ = 0. \tag{8.5}$$

The second of these equations implies that the active fibres lie along geodesic curves on the deformed surface. This result, together with (8.5c), is used to analyse the deformations of half-slack nets in §11.

Suppose traction is specified on a part ∂S_t of the boundary. Let the edges $\boldsymbol{x}(\partial S_t)$ and $\boldsymbol{r}(\partial S_t)$ be parametrized by the arc length parameters s and u, respectively, and suppose the parametrizations are similar in the sense that $\mathrm{d}s/\mathrm{d}u > 0$. Then on ∂S_t we have $\mathrm{d}\boldsymbol{x} = \boldsymbol{s}\,\mathrm{d}s$ and $\mathrm{d}\boldsymbol{r} = \boldsymbol{u}\,\mathrm{d}u$, where $\boldsymbol{s}$ and $\boldsymbol{u}$ are the unit tangents to the edges of the reference and deformed surfaces. It follows from (3.1), (3.2) that

$$u^\alpha = (\mathrm{d}s/\mathrm{d}u)\, s^\alpha, \tag{8.6}$$

where $u^\alpha = \boldsymbol{u}\cdot\boldsymbol{a}^\alpha$ and $s^\alpha = \boldsymbol{s}\cdot\boldsymbol{A}^\alpha$. Let $\boldsymbol{\nu} = \nu_\alpha \boldsymbol{A}^\alpha$ and $\boldsymbol{\omega} = \omega_\alpha \boldsymbol{a}^\alpha$ be the rightward unit normals to the edges. Then $\boldsymbol{\nu} = \boldsymbol{s}\times\boldsymbol{N}$ and $\boldsymbol{\omega} = \boldsymbol{u}\times\boldsymbol{n}$, i.e.

$$\nu_\beta = \mu_{\beta\alpha} s^\alpha, \quad \omega_\beta = \epsilon_{\beta\alpha} u^\alpha. \tag{8.7}$$

Combining these results we get

$$\omega_\beta = (\mathrm{d}s/\mathrm{d}u)\, J\nu_\beta. \tag{8.8}$$

Then from (3.5), (3.14) and (4.19) we can find an expression for the projection of the traction $\boldsymbol{t}$ onto $\boldsymbol{\omega}$:

$$\boldsymbol{t}\cdot\boldsymbol{\omega} = (\mathrm{d}s/\mathrm{d}u)\, J[\lambda^{-1}f(\lambda)\,(\boldsymbol{L}\cdot\boldsymbol{\nu})^2 + \mu^{-1}g(\mu)\,(\boldsymbol{M}\cdot\boldsymbol{\nu})^2]. \tag{8.9}$$

Since the relaxed fibre stresses are non-negative it follows that $\boldsymbol{t}\cdot\boldsymbol{\omega} \geqslant 0$. For a given traction, this inequality places a restriction on the edge deformation.

From the form of the strain energy function adopted here it is evident that there is no energetic penalty associated with local collapse ($\cos\gamma = 0$) of the fibres. Thus it may happen that in equilibrium the fibres of the net collapse in finite regions or on isolated curves in the (θ^1, θ^2) plane. Collapse of fibres on isolated curves is associated with folding of the membrane along the envelopes of the $\boldsymbol{l}$- and $\boldsymbol{m}$-trajectories. These envelopes are the curves on which the condition $\boldsymbol{l}\times\boldsymbol{m} = \boldsymbol{0}$ is satisfied. Then according to (4.14), the limiting tangent planes on either side of an envelope will be oppositely oriented when $\boldsymbol{l}$ and $\boldsymbol{m}$ are continuous. Such folding can therefore occur if the deformation $\boldsymbol{r}(\theta^1, \theta^2)$ is a perfectly smooth function. Accordingly, the analysis of equilibrium for such deformations requires no special consideration of envelopes. Green & Shi (1989) have obtained a solution containing an envelope for the special case of plane deformation.

The treatment of finite regions of fibre collapse is more involved. Suppose that such a region collapses onto the curve $\boldsymbol{r}(t)$, where $t(\theta^1, \theta^2)$ measures arc length along the curve. Then from (4.4) we get

$$\lambda\boldsymbol{l} = (\boldsymbol{L}\cdot\nabla t)\,\boldsymbol{r}'(t), \quad \mu\boldsymbol{m} = (\boldsymbol{M}\cdot\nabla t)\,\boldsymbol{r}'(t). \tag{8.10}$$

Because $\boldsymbol{r}'(t)$ is a unit vector, it follows that

$$\lambda = |\boldsymbol{L}\cdot\nabla t| \quad \text{and} \quad \mu = |\boldsymbol{M}\cdot\nabla t|. \tag{8.11}$$

Then (4.22) furnishes the stress

$$\boldsymbol{T}^\alpha = h^\alpha \boldsymbol{r}'(t); \quad h^\alpha = \lambda^{-1}(\boldsymbol{L}\cdot\nabla t) f(\lambda)\, L^\alpha + \mu^{-1}(\boldsymbol{M}\cdot\nabla t)\, g(\mu)\, M^\alpha. \tag{8.12}$$

The equilibrium equation (3.13) (with zero pressure) reduces to

$$(\nabla\cdot\boldsymbol{h})\,\boldsymbol{r}'(t) + (\boldsymbol{h}\cdot\nabla t)\,\boldsymbol{r}''(t) = \boldsymbol{0}, \tag{8.13}$$

where $\boldsymbol{h} = h^\alpha \boldsymbol{A}_\alpha$. It follows from (8.11) and (8.12b) that

$$\boldsymbol{h}\cdot\nabla t = \lambda f(\lambda) + \mu g(\mu), \tag{8.14}$$

and this is positive unless the net is slack. Then (8.13) is equivalent to the system

$$\boldsymbol{r}''(t) = \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{h} = 0. \tag{8.15}$$

The first of these results implies that the net collapses onto a straight line. The stress in this line is described by (8.15*b*). From (8.12*b*) it is evident that the individual fibre stresses cannot be determined in a collapsed region unless the net is half-slack. In the latter case (8.15*b*) provides a single equation for the active fibre stress. These conclusions are similar to those obtained by Pipkin (1980) in a study of the mechanics of inextensible networks.

9. Weak universal deformations

In this section and the next we attempt to characterize all of the deformations that can be produced in every homogeneous elastic network by application of edge traction and lateral pressure alone, irrespective of the forms of the fibre strain energy functions. Such deformations are said to be universal. The importance of universal deformations stems from the essential role that they play in the experimental determination of the strain energy functions.

The search for universal deformations in the context of the conventional theory of elasticity for isotropic materials was initiated by Ericksen (1954, 1955). Subsequent contributions to the subject are summarized in a review article by Beatty (1987). Detailed analyses of universal deformations of homogeneous, isotropic elastic membranes have been presented by Naghdi & Tang (1977), Wang & Cross (1977) and Yin (1981). Steigmann (1990) has recently enlarged this family of universal solutions by accounting for the possibility of wrinkling of the membrane. Green & Shi (1989) have derived a class of plane universal deformations in the context of the present theory of elastic nets.

The stresses in a net, and in membranes in general, depend directly on the strain as measured by (3.4). This leads to equilibrium equations for a net under lateral pressure whose tangential components do not involve normal curvature, torsion and pressure explicitly (see (8.3)). It follows that an equilibrium strain distribution satisfying (8.3*b*, *c*) can be produced in a net by stretching it over a smooth rigid surface, provided that the normal curvatures and torsion induced in the fibres, together with the fibre stretches and shear angle, satisfy the Gauss and Mainardi–Codazzi equations. Contact with this surface then furnishes whatever distribution of pressure that may be required to maintain equilibrium in the normal direction. A similar observation in the context of isotropic membrane theory led Yin (1981) to classify universal deformations as weak or strong. The weak universal deformations involve restrictions on the strain arising only from the tangential equations of equilibrium. Strong universal deformations are further restricted by the requirement that the normal equation of equilibrium be satisfied for a particular distribution of pressure. We study the weak solutions for tense nets in the present section. Strong solutions are obtained in §10 for the case of uniformly distributed pressure.

(*a*) *Tangential equilibrium*

The tangential equilibrium of a net is described by (8.3*b*, *c*) or, equivalently, by the projection of (8.2) onto the tangent plane. According to (5.22), this can be expressed in the form

$$f'(\lambda)\,(\boldsymbol{L}\cdot\nabla\lambda)\,\boldsymbol{l}+g'(\mu)\,(\boldsymbol{M}\cdot\nabla\mu)\,\boldsymbol{m}+f(\lambda)\,(\lambda\eta_l\boldsymbol{p}+\eta_M\boldsymbol{l})+g(\mu)\,(\mu\eta_m\boldsymbol{q}-\eta_L\boldsymbol{m})=\boldsymbol{0} \tag{9.1}$$

for fibre strain energies that do not depend explicitly on θ^1 and θ^2. If (9.1) is to be satisfied for every choice of the fibre response functions, then it is necessary and sufficient that the coefficients of f, g, f' and g' vanish separately:

$$\boldsymbol{L}\cdot\nabla\lambda = 0, \quad \boldsymbol{M}\cdot\nabla\mu = 0 \tag{9.2}$$

and

$$\lambda\eta_l\,\boldsymbol{p}+\eta_M\,\boldsymbol{l} = \boldsymbol{0}, \quad \mu\eta_m\,\boldsymbol{q}-\eta_L\,\boldsymbol{m} = \boldsymbol{0}. \tag{9.3}$$

Since $\{\boldsymbol{l},\boldsymbol{p}\}$ and $\{\boldsymbol{m},\boldsymbol{q}\}$ span the tangent plane it follows that

$$\eta_L = 0, \quad \eta_M = 0; \quad \eta_l = 0, \quad \eta_m = 0. \tag{9.4}$$

From (9.2) it follows that the fibre stretches are constant along their respective fibres, i.e. each fibre is uniformly stretched. According to (9.4), both families of fibres lie along geodesics on the reference and deformed surfaces. Furthermore, (9.4a,b) and (5.36a) imply that the gaussian curvature of the reference surface vanishes. Thus according to a famous theorem, the reference surface, if sufficiently smooth, is developable onto the plane, at least locally (Kreyszig 1968, §52). The images of the geodesics are straight lines in the plane.

Since the stresses are bending invariants we can take the $\boldsymbol{L}$- and $\boldsymbol{M}$-trajectories to lie along rectangular coordinates x and y in the plane without loss of generality. Then equations (9.2) reduce to

$$\partial\lambda/\partial x = 0, \quad \partial\mu/\partial y = 0, \tag{9.5}$$

and with the aid of (5.21) we find that (9.4c,d) become

$$\partial(\mu\sin\gamma)/\partial x = \partial\lambda/\partial y, \quad \partial(\lambda\sin\gamma)/\partial y = \partial\mu/\partial x. \tag{9.6}$$

It follows from (9.5) that λ and μ are functions of y and x, respectively, and from (9.6) that

$$\mu(x)\sin\gamma = x\lambda'(y)+a(y), \quad \lambda(y)\sin\gamma = y\mu'(x)+b(x), \tag{9.7}$$

where $a(y)$ and $b(x)$ are arbitrary functions. Eliminating $\sin\gamma$, we obtain

$$x\lambda\lambda'+\lambda a = y\mu\mu'+\mu b. \tag{9.8}$$

To find necessary conditions for this we assume sufficient smoothness and differentiate twice with respect to x:

$$y(\mu\mu')''+(\mu b)'' = 0. \tag{9.9}$$

Differentiation of this with respect to y yields $(\mu\mu')'' = 0$, from which we obtain

$$\mu^2 = A_1x^2+2B_1x+C_1, \tag{9.10}$$

where A_1, B_1 and C_1 are constants. Now (9.9) gives $(\mu b)'' = 0$ and therefore

$$b = \mu^{-1}(D_1x+E_1), \tag{9.11}$$

where D_1 and E_1 are constants. Further necessary conditions for (9.8) are obtained by differentiating with respect to y instead of x. This leads to

$$\lambda^2 = A_2y^2+2B_2y+C_2, \quad a = \lambda^{-1}(D_2y+E_2), \tag{9.12}$$

where A_2, B_2, C_2, D_2 and E_2 are constants. These constants are not independent. Substituting (9.10)–(9.12) back into (9.8), we find

$$x(A_2y+B_2)+D_2y+E_2 = y(A_1x+B_1)+D_1x+E_1, \tag{9.13}$$

from which we conclude that

$$A_1 = A_2 (\equiv A), \quad B_1 = D_2, \quad B_2 = D_1 \quad \text{and} \quad E_1 = E_2 (\equiv E). \tag{9.14}$$

The fibre shear angle can then be found from either of equations (9.7):

$$\sin\gamma = (\lambda\mu)^{-1}(Axy + B_2 x + B_1 y + E). \tag{9.15}$$

(*b*) *Identical, equally stretched fibres*

Additional solutions can be obtained for nets in which both families of fibres have identical response functions, i.e. $g(\cdot) = f(\cdot)$. If the fibres are equally stretched ($\mu = \lambda$) then (9.1) becomes

$$f'(\lambda)\,[(\boldsymbol{L}\cdot\nabla\lambda)\,\boldsymbol{l} + (\boldsymbol{M}\cdot\nabla\lambda)\,\boldsymbol{m}] + f(\lambda)\,(\lambda\eta_l\,\boldsymbol{p} + \eta_M\,\boldsymbol{l} + \lambda\eta_m\,\boldsymbol{q} - \eta_L\,\boldsymbol{m}) = \boldsymbol{0}. \tag{9.16}$$

The deformation is universal if and only if

$$\left.\begin{aligned} (\boldsymbol{L}\cdot\nabla\lambda)\,\boldsymbol{l} + (\boldsymbol{M}\cdot\nabla\lambda)\,\boldsymbol{m} &= \boldsymbol{0}, \\ \lambda\eta_l\,\boldsymbol{p} + \eta_M\,\boldsymbol{l} + \lambda\eta_m\,\boldsymbol{q} - \eta_L\,\boldsymbol{m} &= \boldsymbol{0}. \end{aligned}\right\} \tag{9.17}$$

From (9.17*a*) we have $\boldsymbol{L}\cdot\nabla\lambda = 0$, $\boldsymbol{M}\cdot\nabla\lambda = 0$ and these imply that $\nabla\lambda = \boldsymbol{0}$, i.e.

$$\lambda = \text{const.} \tag{9.18}$$

Now we scalar-multiply (9.17*b*) by $\boldsymbol{l}$ and $\boldsymbol{m}$ to obtain

$$\left.\begin{aligned} \eta_M - \lambda\eta_m|\cos\gamma| - \eta_L \sin\gamma &= 0, \\ -\eta_L + \lambda\eta_l|\cos\gamma| + \eta_M \sin\gamma &= 0. \end{aligned}\right\} \tag{9.19}$$

According to (4.12), (5.21), (5.22) and (9.18),

$$\left.\begin{aligned} \lambda\eta_l|\cos\gamma| &= \boldsymbol{L}\cdot\nabla(\sin\gamma) + \eta_M \sin\gamma + \eta_L, \\ \lambda\eta_m|\cos\gamma| &= \eta_M - \boldsymbol{M}\cdot\nabla(\sin\gamma) + \eta_L \sin\gamma. \end{aligned}\right\} \tag{9.20}$$

Then (9.19) furnishes the system

$$\boldsymbol{L}\cdot\nabla(\sin\gamma) = -2\eta_M \sin\gamma, \quad \boldsymbol{M}\cdot\nabla(\sin\gamma) = 2\eta_L \sin\gamma, \tag{9.21}$$

which is equivalent to

$$\nabla(\sin\gamma) = 2\sin\gamma(\eta_L\,\boldsymbol{M} - \eta_M\,\boldsymbol{L}). \tag{9.22}$$

Clearly $\sin\gamma \equiv 0$ is a solution and this, together with (9.18) and (4.10), implies that the strain is a uniform dilation: $a_{\alpha\beta} = \lambda^2 A_{\alpha\beta}$. In this case there are no restrictions on the reference surface.

If $\sin\gamma$ is not identically zero then (9.22) is equivalent to

$$\nabla(\ln|\sin\gamma|^{\frac{1}{2}}) = \eta_L\,\boldsymbol{M} - \eta_M\,\boldsymbol{L}. \tag{9.23}$$

Then if the right-hand side is continuously differentiable we require

$$\boldsymbol{N}\cdot\nabla\times(\eta_L\,\boldsymbol{M} - \eta_M\,\boldsymbol{L}) = 0. \tag{9.24}$$

This is a restriction on the initial fibre arrangement that is necessary for existence of a universal solution. If this condition is satisfied then $|\sin\gamma|$ is determined from (9.23) by integration. Equation (9.24) implies the existence of a twice continuously differentiable function $\phi(\theta^1, \theta^2)$ such that

$$\eta_L = \boldsymbol{M}\cdot\nabla\phi, \quad \eta_M = -\boldsymbol{L}\cdot\nabla\phi, \tag{9.25}$$

and (9.23) then gives

$$\sin\gamma = c\,\mathrm{e}^{2\phi}, \tag{9.26}$$

where c is an arbitrary constant.

To interpret (9.25) we can make use of the fibre coordinates ξ and η introduced in (7.11). The associated metric components are $(A_{\alpha\beta}) = \mathrm{diag}\,(\alpha^2, \beta^2)$ (see (7.11)) and the components of $\boldsymbol{L}$ and $\boldsymbol{M}$ are $L^1 = \alpha^{-1}$, $L^2 = 0$, $M^1 = 0$ and $M^2 = \beta^{-1}$. Then equations (9.25) become

$$\eta_L = \beta^{-1}\,\partial\phi/\partial\eta, \quad \eta_M = -\alpha^{-1}\,\partial\phi/\partial\xi. \tag{9.27}$$

From (5.22) we have

$$\eta_L = -(\alpha\beta)^{-1}\,\partial\alpha/\partial\eta \quad \text{and} \quad \eta_M = (\alpha\beta)^{-1}\,\partial\beta/\partial\xi \tag{9.28}$$

and therefore, since α and β are positive by definition,

$$\partial(\ln\beta)/\partial\xi = -\partial\phi/\partial\xi, \quad \partial(\ln\alpha)/\partial\eta = -\partial\phi/\partial\eta. \tag{9.29}$$

These give

$$\alpha = a(\xi)\,\mathrm{e}^{-\phi} \quad \text{and} \quad \beta = b(\eta)\,\mathrm{e}^{-\phi}, \tag{9.30}$$

where $a(\xi)$ and $b(\eta)$ are arbitrary positive functions. Thus the first fundamental form on the reference surface must be expressible as

$$|\mathrm{d}\boldsymbol{x}|^2 = [A(\xi)\,\mathrm{d}\xi^2 + B(\eta)\,\mathrm{d}\eta^2]\,\rho(\xi,\eta); \quad \rho = \mathrm{e}^{-2\phi}, \tag{9.31}$$

where $A = a^2$ and $B = b^2$. Coordinates that yield a first fundamental form of this type are known as isometric parameters (the terms isothermal and isothermic are also used). Thus we find that the fibre trajectories must be isometric curves on the reference surface. This does not seriously restrict the class of admissible reference surfaces, for it is known that every sufficiently smooth surface can be parametrized, at least locally, in terms of isometric coordinates (Kreyszig 1968, §58). Simple examples of isometric curves are the meridians and circles of latitude on surfaces of revolution and the lines of curvature on minimal surfaces (see, for example, Weatherburn 1961, §§39 and 91). If the initial fibre trajectories are isometric curves as required, then there exist universal solutions with $\sin\gamma$ given by (9.26) and (9.31b):

$$\sin\gamma = c/\rho(\xi,\eta). \tag{9.32}$$

To summarize, weak universal deformations of arbitrary nets exist only if the reference surface is developable onto the plane. The distributions of the stretches and shear angle are given by (9.10), (9.12a), (9.14) and (9.15) in terms of plane rectangular coordinates. If the net consists of identical fibre families and these are equally stretched, then the stretch is uniform. Uniform surface dilations with $\sin\gamma$ vanishing identically can be produced in every such net. Solutions with $\sin\gamma$ not zero exist only if the fibres lie along isometric curves on the reference surface. Then $\sin\gamma$ is determined by the geometry of these curves, apart from an arbitrary constant.

10. Strong universal deformations

Strong universal deformations are those weak deformations that can be maintained in every net by a lateral pressure whose distribution is prescribed. Here we consider only uniform pressure, including the special case of zero pressure.

(a) *Arbitrary nets*

The equation of normal equilibrium is obtained from scalar multiplication of (8.2) by $\boldsymbol{n}$:

$$\lambda f(\lambda)\,\kappa_l + \mu g(\mu)\,\kappa_m + pJ = 0. \tag{10.1}$$

If the pressure vanishes then we can obtain the deformation explicitly: In this case (10.1) is satisfied for every choice of the fibre response functions if and only if κ_l and κ_m vanish. These results, together with (9.4*c*, *d*), (9.5), (5.14*a*, *b*) and (4.6) yield

$$\partial(\lambda \boldsymbol{l})/\partial x = \boldsymbol{0}, \quad \partial(\mu \boldsymbol{m})/\partial y = \boldsymbol{0}. \tag{10.2}$$

According to (4.4), these are equivalent to $\partial^2 \boldsymbol{r}/\partial x^2 = \boldsymbol{0}$ and $\partial^2 \boldsymbol{r}/\partial y^2 = \boldsymbol{0}$, respectively, and therefore

$$\boldsymbol{r} = \boldsymbol{A}xy + \boldsymbol{B}x + \boldsymbol{C}y + \boldsymbol{D}, \tag{10.3}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are arbitrary constant vectors.

Conversely, if the deformation is given by (10.3) then (9.4*c*, *d*) and (9.5) are satisfied and it follows that (9.10), (9.12*a*), (9.14) and (9.15) are valid. To see this we compute the deformation gradient from (4.4, 7):

$$\boldsymbol{F} = (\partial \boldsymbol{r}/\partial x) \otimes \boldsymbol{L} + (\partial \boldsymbol{r}/\partial y) \otimes \boldsymbol{M} = (\boldsymbol{A}y + \boldsymbol{B}) \otimes \boldsymbol{L} + (\boldsymbol{A}x + \boldsymbol{C}) \otimes \boldsymbol{M}. \tag{10.4}$$

This delivers the strain in the form (4.9), where

$$\left.\begin{aligned} \lambda^2 = \boldsymbol{A}\cdot\boldsymbol{A}y^2 + 2\boldsymbol{A}\cdot\boldsymbol{B}y + \boldsymbol{B}\cdot\boldsymbol{B}, \quad \mu^2 = \boldsymbol{A}\cdot\boldsymbol{A}x^2 + 2\boldsymbol{A}\cdot\boldsymbol{C}x + \boldsymbol{C}\cdot\boldsymbol{C} \\ \text{and} \qquad \lambda\mu \sin\gamma = \boldsymbol{A}\cdot\boldsymbol{A}xy + \boldsymbol{A}\cdot\boldsymbol{B}x + \boldsymbol{A}\cdot\boldsymbol{C}y + \boldsymbol{B}\cdot\boldsymbol{C}. \end{aligned}\right\} \tag{10.5}$$

These are the same as (9.10), (9.12*a*), (9.14) and (9.15), with

$$A = \boldsymbol{A}\cdot\boldsymbol{A}, \quad B_2 = \boldsymbol{A}\cdot\boldsymbol{B}, \quad B_1 = \boldsymbol{A}\cdot\boldsymbol{C}, \quad C_1 = \boldsymbol{C}\cdot\boldsymbol{C}, \quad C_2 = \boldsymbol{B}\cdot\boldsymbol{B}, \quad E = \boldsymbol{B}\cdot\boldsymbol{C}. \tag{10.6}$$

This solution contains the plane universal deformation of Green & Shi (1989) as a special case. In the general case, it follows from (5.8) and (5.14*c*, *d*) that (10.3) describes a surface with variable, non-positive gaussian curvature.

For non-zero pressures we write (10.1) as

$$-p = J^{-1}[A(\lambda)\,\kappa_l + B(\mu)\,\kappa_m], \tag{10.7}$$

where

$$A(\lambda) \equiv \lambda f(\lambda), \quad B(\mu) \equiv \mu g(\mu). \tag{10.8}$$

We assume that the pressure is uniform, i.e. $\nabla p = \boldsymbol{0}$. Then from (10.7) and (9.5) we obtain

$$\begin{aligned} A(\lambda)\,(J\nabla\kappa_l - \kappa_l \nabla J) + B(\mu)\,(J\nabla\kappa_m - \kappa_m \nabla J) \\ + A'(\lambda)\,J\kappa_l\,\lambda'(y)\,\nabla y + B'(\mu)\,J\kappa_m\,\mu'(x)\,\nabla x = \boldsymbol{0}. \end{aligned} \tag{10.9}$$

This is satisfied for every choice of the fibre response functions if and only if

$$\left.\begin{aligned} \nabla(\ln|\kappa_l|) = \nabla(\ln|\kappa_m|) = \nabla(\ln J) \\ \text{and} \qquad \kappa_l\,\lambda'(y) = 0, \quad \kappa_m\,\mu'(x) = 0, \end{aligned}\right\} \tag{10.10}$$

and therefore

$$\kappa_l = aJ, \quad \kappa_m = bJ, \quad a\lambda'(y) = 0, \quad b\mu'(x) = 0, \tag{10.11}$$

where a and b are arbitrary constants.

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If a and b both vanish then the pressure vanishes also. We have already considered this case. The remaining possibilities are: (i) a and b are non-zero, (ii) either a or b vanishes. To consider these cases further we use (9.4c,d) to reduce the Gauss equation (5.34) to the form

$$J^2\kappa = -(\lambda\mu\cos\gamma)\,\partial^2\gamma/\partial x\,\partial y. \tag{10.12}$$

From (5.35) and (9.4) the Mainardi–Codazzi equations become

$$\left.\begin{aligned} J(\lambda\mu)^{-2}[\partial(\mu\tau)/\partial x-\partial(\lambda\kappa_l)/\partial y] &= \phi_l(\kappa_l\sin\gamma-\tau),\\ J(\lambda\mu)^{-2}[\partial(\mu\kappa_m)/\partial x-\partial(\lambda\tau)/\partial y] &= \phi_m(\kappa_m\sin\gamma-\tau),\end{aligned}\right\} \tag{10.13}$$

where, in consequence of (5.27) and (5.28),

$$J\phi_l = (\lambda\cos\gamma)\,\partial\gamma/\partial y,\quad J\phi_m = -(\mu\cos\gamma)\,\partial\gamma/\partial x. \tag{10.14}$$

These combine to give

$$\left.\begin{aligned} (\cos\gamma)\,[\partial(\mu\tau)/\partial x-\partial(\lambda\kappa_l)/\partial y] &= \lambda(\kappa_l\sin\gamma-\tau)\,\partial\gamma/\partial y,\\ (\cos\gamma)\,[\partial(\lambda\tau)/\partial y-\partial(\mu\kappa_m)/\partial x] &= \mu(\kappa_m\sin\gamma-\tau)\,\partial\gamma/\partial x.\end{aligned}\right\} \tag{10.15}$$

If neither of the constants a, b vanishes, then (10.11 c,d) imply that λ and μ are constants. We conclude from (9.10), (9.12a), (9.14) and (9.15) that $\sin\gamma$ is also constant. Then $\kappa = 0$ according to (10.12), i.e. the deformed surface is developable onto the plane. From (10.11 a,b) we also infer that κ_l and κ_m are constants, and then from (5.8) that τ is constant. This solution satisfies the Gauss and Mainardi–Codazzi equations identically. Now suppose that $b = 0$ and $a \neq 0$. According to (10.11 c) we have $\lambda = \text{const}$. This result, together with (9.12) and (9.14), implies that $A = 0$ and $B_2 = 0$. Then (9.10), (9.15), (10.11) and (5.8) can be used to put the Gauss equation (10.12) into the form $\tau^2 = -B_1^2\sec^2\gamma(\lambda\mu^2)^{-2}$, whence it follows that $B_1 = 0$ and $\tau = 0$. Thus μ, $\sin\gamma$, κ_l are constants and $\kappa = 0$, so that the deformed surface is again developable onto the plane and the Mainardi–Codazzi equations are trivially satisfied. The case $a = 0$ and $b \neq 0$ is similar.

Thus we find that strong universal deformations of arbitrary nets under uniform pressure are constant-strain mappings of developable surfaces onto developable surfaces. The fibres are mapped onto geodesics of constant torsion forming either straight lines or circular arcs that lie in planes containing the surface normal. If the pressure vanishes then the deformation maps developable surfaces onto the surfaces of variable, non-positive gaussian curvature described by (10.3). The fibres are mapped onto straight lines.

(b) *Identical, equally stretched fibres*

For nets composed of identical fibres that are equally stretched, (10.7) reduces to

$$-p = J^{-1}A(\lambda)\,(\kappa_l+\kappa_m). \tag{10.16}$$

Then from (9.18) it follows that (10.16) is satisfied for all such nets under uniform pressure if and only if

$$\kappa_l+\kappa_m = aJ;\quad a = \text{const}. \tag{10.17}$$

In view of the analysis of §9, there are two cases to consider according as $\sin\gamma$ is identically zero or not. In the first instance we find from (9.18), (5.21), (5.22) and (5.27) that

$$\lambda\eta_l = \eta_L,\quad \lambda\eta_m = \eta_M;\quad \phi_l = \eta_m,\quad \phi_m = \eta_l. \tag{10.18}$$

Then the Gauss equations (5.34) and (5.36 a) give

$$\lambda^2\kappa = (\eta_L M^\alpha - \eta_M L^\alpha)|_\alpha = \bar{\kappa}, \tag{10.19}$$

and the Mainardi–Codazzi equations (5.35) become

$$\left.\begin{aligned}(\tau L^\alpha - \kappa_l M^\alpha)|_\alpha &= \eta_L \kappa_m - \eta_M \tau,\\ (\kappa_m L^\alpha - \tau M^\alpha)|_\alpha &= \eta_M \kappa_l - \eta_L \tau.\end{aligned}\right\} \tag{10.20}$$

Recalling (5.36 b, c), we see that these are also the equations that are satisfied by the fibre curvatures and torsion on the reference surface.

A fundamental existence and uniqueness theorem in surface theory (see, for example, Stoker 1969, ch. 6) states that a solution of the Gauss and Mainardi–Codazzi equations determines a surface uniquely apart from rigid body motions. Conversely, a surface and all other surfaces obtained from it by a rigid motion furnish unique values of $b_{\alpha\beta}$ for a given choice of convected coordinates. From this theorem and equations (10.19), (10.20), (5.8) and (5.37) it follows that

$$\kappa_l = \pm\lambda^{-1}\kappa_L, \quad \kappa_m = \pm\lambda^{-1}\kappa_M, \quad \tau = \pm\lambda^{-1}\bar{\tau}. \tag{10.21}$$

Now from (10.17) the mean curvature of the deformed surface is constant and (10.21 a, b) then imply that the mean curvature of the reference surface is also constant. If the membrane is closed then both the reference and deformed surfaces are spheres (Hopf 1989, ch. 6). If the pressure vanishes then both surfaces are isometric to minimal surfaces. In either case the strain consists of a uniform surface dilation.

If the fibres lie along isometric curves on the reference surface then $\sin\gamma$ need not vanish identically. In this case $\sin\gamma$ is determined by the geometry of these curves in accordance with (9.32). The analysis of the Gauss and Mainardi–Codazzi equations therefore depends on the choice of reference surface. For illustrative purposes we consider fibres that initially lie along rectangular coordinate axes in the plane. Then $\eta_L = \eta_M = 0$ and from (9.19) we find that η_l and η_m vanish, i.e. the deformed fibres are geodesics. From (9.22) (or (9.32)) it also follows that $\sin\gamma = \text{const.}$ With these results, the Gauss equation (5.34) gives $\kappa = 0$, i.e.

$$\tau^2 = \kappa_l \kappa_m. \tag{10.22}$$

In the present context the Mainardi–Codazzi equations (5.35) take the simple forms

$$\partial\tau/\partial x = \partial\kappa_l/\partial y, \quad \partial\kappa_m/\partial x = \partial\tau/\partial y, \tag{10.23}$$

where x and y are the fibre coordinates. From (10.17) it also follows that

$$\partial\kappa_l/\partial x = -\partial\kappa_m/\partial x, \quad \partial\kappa_l/\partial y = -\partial\kappa_m/\partial y \tag{10.24}$$

and these imply that κ_l, κ_m and τ are harmonic, i.e.

$$\tau + \mathrm{i}\kappa_l = a(z), \quad \kappa_m + \mathrm{i}\tau = b(z) \quad \text{and} \quad \kappa_l + \mathrm{i}\kappa_m = c(z), \tag{10.25}$$

where a, b, c are analytic functions of the complex variable $z = x + \mathrm{i}y$, subject to the restrictions

$$\mathrm{Re}\,(a) = \mathrm{Im}\,(b), \quad \mathrm{Im}\,(a) = \mathrm{Re}\,(c), \quad \mathrm{Re}\,(b) = \mathrm{Im}\,(c). \tag{10.26}$$

Now (10.17) implies that $\mathrm{Re}\,(c) + \mathrm{Im}\,(c) = \text{const.}$ and since $c(z)$ is analytic it follows that $c(z) = \text{const.}$ Then $a(z)$ and $b(z)$ are also constants from (10.26). This in turn implies that κ_l, κ_m and τ are all constants, with τ determined in terms of κ_l and κ_m by (10.22).

Thus the deformation is a constant-strain mapping of the plane onto a developable surface. The fibres are mapped onto geodesics of constant curvature and torsion. If the pressure is zero then (10.16) gives $\kappa_m = -\kappa_l$, (10.22) gives $\tau^2 = -\kappa_l^2$, and these imply that $\kappa_l = 0$, $\kappa_m = 0$ and $\tau = 0$. Thus the curvature tensor vanishes and the deformed surface is a plane.

11. Equilibrium of half-slack nets

(a) *General theory*

The equilibrium theory of half-slack nets has certain features that distinguish it from the theory of tense nets considered thus far. In particular, we find that the deformation can be evaluated explicitly in the absence of lateral pressure. Here we consider deformations for which $\mu \in [0, 1]$ and $\lambda > 1$. (The treatment of the case $\lambda \in [0, 1]$ and $\mu > 1$ is similar and is therefore omitted.) Then $g(\mu) \equiv 0$ and equilibrium configurations of the net are described by (8.5). The results (8.5 *b*, *c*) lead to an equation for the deformation $\boldsymbol{r}(\theta^1, \theta^2)$. From (5.14 *a*) and (4.6 *a*),

$$\lambda^{-1}(\boldsymbol{L}\cdot\nabla)\,\boldsymbol{l} = -p(\lambda f)^{-1} J\boldsymbol{n}, \tag{11.1}$$

and (4.12) and (4.14) provide

$$J\boldsymbol{n} = \lambda\mu\boldsymbol{l}\times\boldsymbol{m}. \tag{11.2}$$

These combine with (4.4) to give

$$\lambda^{-1}(\boldsymbol{L}\cdot\nabla)\,[\lambda^{-1}(\boldsymbol{L}\cdot\nabla)\,\boldsymbol{r}] = -p[\lambda f(\lambda)]^{-1}(\boldsymbol{L}\cdot\nabla)\,\boldsymbol{r}\times(\boldsymbol{M}\cdot\nabla)\,\boldsymbol{r}. \tag{11.3}$$

This result and (8.5 *a*),

$$\nabla\cdot[f(\lambda)\,\boldsymbol{L}] = 0, \tag{11.4}$$

furnish the equilibrium equations for half-slack nets.

Suppose the traction $\boldsymbol{t}$ is specified on a part ∂S_t of the boundary. Then from (3.14) and (4.21) (with $g = 0$) we have

$$\boldsymbol{t} = f(\lambda)\,(\boldsymbol{L}\cdot\boldsymbol{\nu})\,\boldsymbol{l}; \quad (\theta^1, \theta^2) \in \partial S_t, \tag{11.5}$$

where $\boldsymbol{\nu}$ is the rightward unit normal to the arc $\boldsymbol{x}(\partial S_t)$. Since $f > 0$ we require

$$f(\lambda)\,|\boldsymbol{L}\cdot\boldsymbol{\nu}| = |\boldsymbol{t}|, \quad \operatorname{sgn}(\boldsymbol{L}\cdot\boldsymbol{\nu})\,\boldsymbol{l} = \boldsymbol{t}/|\boldsymbol{t}|; \quad (\theta^1, \theta^2) \in \partial S_t, \tag{11.6}$$

and thus traction data deliver the boundary values of f and $\boldsymbol{l}$. It follows from (11.4) that the traction problem is statically determinate in the sense that f is independent of the deformation. If $\boldsymbol{t}$ vanishes on the boundary then either $\boldsymbol{L}\cdot\boldsymbol{\nu} = 0$, so that ∂S_t is an $\boldsymbol{L}$-trajectory, or $f = 0$ and the boundary values of $\boldsymbol{l}$ can be chosen arbitrarily. Because (11.3) and (11.4) is not an elliptic system, existence of solutions to the traction problem is not to be expected unless the data are suitably restricted. A similar remark applies to the problem of placement in which $\boldsymbol{r}$ is prescribed on a part ∂S_r of the boundary.

For pressurized nets the solution of (11.3) presents a formidable challenge. Nevertheless, we can use (11.4) and (8.5 *b*) to obtain information about the distribution of strain in the net without recourse to (11.3). According to (5.21 *a*) and (2.20 *b*), we can write (8.5 *b*) in the form

$$\boldsymbol{N}\cdot\nabla\times(\lambda\boldsymbol{L} + \mu\sin\gamma\boldsymbol{M}) = 0. \tag{11.7}$$

The system (11.4), (11.7) furnishes restrictions on λ and the combination $\mu\sin\gamma$.

However, (11.7) is automatically satisfied if $\boldsymbol{r}$ satisfies (11.3) and is therefore redundant if an expression for the deformation is available. To prove this claim we introduce a coordinate transformation $(\theta^1, \theta^2) \rightarrow [\phi(\theta^1, \theta^2), \psi(\theta^1, \theta^2)]$ defined by

$$\boldsymbol{L}\cdot\nabla\phi = \lambda, \quad \boldsymbol{L}\cdot\nabla\psi = 0. \tag{11.8}$$

Then the gradients of ϕ and ψ are

$$\nabla\phi = \lambda\boldsymbol{L} + (\boldsymbol{M}\cdot\nabla\phi)\,\boldsymbol{M}, \quad \nabla\psi = (\boldsymbol{M}\cdot\nabla\psi)\,\boldsymbol{M} \tag{11.9}$$

and the active fibres are the curves $\boldsymbol{M}\cdot \mathrm{d}\boldsymbol{x} = 0$ on which $\psi = \text{const}$. The jacobian of the transformation,

$$\boldsymbol{N}\cdot\nabla\phi\times\nabla\psi = \lambda(\boldsymbol{M}\cdot\nabla\psi), \tag{11.10}$$

is non-zero on the material domain unless ψ is identically constant. Barring this case, the parameters ϕ and ψ define a set of coordinates in the (θ^1, θ^2)-plane and the gradient operator can then be written

$$\nabla = \nabla\phi(\partial/\partial\phi) + \nabla\psi(\partial/\partial\psi). \tag{11.11}$$

This furnishes the directional derivative

$$\lambda^{-1}(\boldsymbol{L}\cdot\nabla) = \partial/\partial\phi; \quad \boldsymbol{l} = \lambda^{-1}(\boldsymbol{L}\cdot\nabla)\,\boldsymbol{r} = \partial\boldsymbol{r}/\partial\phi. \tag{11.12}$$

Thus ϕ measures arc length along the $\boldsymbol{l}$-trajectories. We also find

$$\mu\boldsymbol{m} = (\boldsymbol{M}\cdot\nabla)\,\boldsymbol{r} = (\boldsymbol{M}\cdot\nabla\phi)\,\partial\boldsymbol{r}/\partial\phi + (\boldsymbol{M}\cdot\nabla\psi)\,\partial\boldsymbol{r}/\partial\psi \tag{11.13}$$

and then (11.3) becomes

$$\partial^2\boldsymbol{r}/\partial\phi^2 = -pf^{-1}(\boldsymbol{M}\cdot\nabla\psi)\,(\partial\boldsymbol{r}/\partial\phi)\times(\partial\boldsymbol{r}/\partial\psi). \tag{11.14}$$

If we choose $\boldsymbol{M}\cdot\nabla\psi = f$, so that

$$\nabla\psi = f\boldsymbol{M}, \tag{11.15}$$

then (11.14) reduces to

$$\partial^2\boldsymbol{r}/\partial\phi^2 = -p(\partial\boldsymbol{r}/\partial\phi)\times(\partial\boldsymbol{r}/\partial\psi). \tag{11.16}$$

Such a choice is always possible if $f\boldsymbol{M}$ is smooth and the (θ^1, θ^2)-plane is simply connected, for

$$\boldsymbol{N}\cdot\nabla\times(f\boldsymbol{M}) = 0 \tag{11.17}$$

in view of (11.4).

Under similar restrictions it follows from (11.7) that there exists a twice continuously differentiable function $E(\phi, \psi)$ such that

$$\lambda\boldsymbol{L} + \mu\sin\gamma\boldsymbol{M} = \nabla E = (\partial E/\partial\phi)\,\nabla\phi + (\partial E/\partial\psi)\,\nabla\psi. \tag{11.18}$$

Then $\lambda = (\boldsymbol{L}\cdot\nabla\phi)\,\partial E/\partial\phi$ and from (11.8*a*) we find that $\partial E/\partial\phi = 1$, i.e.

$$E = \phi + H(\psi) \tag{11.19}$$

for some function $H(\cdot)$. Then (11.18) gives

$$\mu\sin\gamma = \boldsymbol{M}\cdot\nabla E = \boldsymbol{M}\cdot\nabla\phi + H'(\psi)\,\boldsymbol{M}\cdot\nabla\psi. \tag{11.20}$$

Alternatively, a solution $\boldsymbol{r}$ of (11.14), together with (4.8) and (11.13), provides

$$\mu\sin\gamma = \boldsymbol{M}\cdot\nabla\phi + (\boldsymbol{M}\cdot\nabla\psi)\,(\partial\boldsymbol{r}/\partial\phi)\cdot(\partial\boldsymbol{r}/\partial\psi). \tag{11.21}$$

Thus we require

$$H'(\psi) = (\partial \boldsymbol{r}/\partial\phi)\cdot(\partial \boldsymbol{r}/\partial\psi). \tag{11.22}$$

This restriction is automatically satisfied by any twice continuously differentiable solution of (11.14). To see this, differentiate the right-hand side of (11.22) with respect to ϕ to get

$$\left(\frac{\partial \boldsymbol{r}}{\partial\phi}\right)\cdot\left(\frac{\partial^2 \boldsymbol{r}}{\partial\phi\,\partial\psi}\right)+\left(\frac{\partial \boldsymbol{r}}{\partial\psi}\right)\cdot\left(\frac{\partial^2 \boldsymbol{r}}{\partial\phi^2}\right) = \boldsymbol{l}\cdot\left(\frac{\partial \boldsymbol{l}}{\partial\psi}\right)-pf^{-1}(\boldsymbol{M}\cdot\nabla\psi)\,\boldsymbol{l}\times\left(\frac{\partial \boldsymbol{r}}{\partial\psi}\right)\cdot\left(\frac{\partial \boldsymbol{r}}{\partial\psi}\right), \tag{11.23}$$

which vanishes identically. Thus (11.7) is not an independent condition.

The formulation involving the coordinates ϕ and ψ is particularly useful if the lateral pressure vanishes. For then (11.14) (or (11.16)) can be integrated immediately to yield

$$\boldsymbol{r} = \phi\boldsymbol{l}(\psi)+\boldsymbol{u}(\psi), \tag{11.24}$$

where $\boldsymbol{u}$ is an arbitrary vector-valued function. This is the equation of a ruled surface (Struik 1961, §5.5) generated by a one-parameter family of unit vectors $\boldsymbol{l}$. The partial differential equations (11.4), (11.8) furnish restrictions on λ, ϕ and ψ which, in conjunction with suitable boundary data, can be used to obtain the deformation explicitly. For example, suppose a fibre $\psi = C$ (const.) intersects a part ∂S_r of a boundary on which $\boldsymbol{r}$ is prescribed. Let A and B be the points of intersection. Then (11.24) gives

$$\boldsymbol{r}(A)-\boldsymbol{r}(B) = [\phi(A)-\phi(B)]\,\boldsymbol{l}(C), \tag{11.25}$$

from which boundary values of ϕ can be deduced.

If the deformation is known we use (11.13) to find

$$\mu\boldsymbol{m} = (\boldsymbol{M}\cdot\nabla\phi)\,\boldsymbol{l}(\psi)+(\boldsymbol{M}\cdot\nabla\psi)\,[\phi\boldsymbol{l}'(\psi)+\boldsymbol{u}'(\psi)], \tag{11.26}$$

which can be used to verify that $\mu\in[0,1]$ *a posteriori* and to locate the curves on which $\mu = 1$ that form the boundaries between tense and half-slack regions of the net. The fibre shear angle then follows from (11.21) and (11.24):

$$\mu\sin\gamma = \boldsymbol{M}\cdot\nabla\phi+(\boldsymbol{M}\cdot\nabla\psi)\,(\boldsymbol{l}\cdot\boldsymbol{u}'). \tag{11.27}$$

(b) *Plane rectangular network*

The foregoing theory is particularly easy to apply if the fibres form a plane rectangular network in the reference configuration. Let $\boldsymbol{i}$ and $\boldsymbol{j}$ be fixed unit vectors in the x and y coordinate directions and suppose $\alpha\in[0,\tfrac{1}{2}\pi)$ is the fixed angle between the $\boldsymbol{L}$-trajectories and the x-axis:

$$\boldsymbol{L} = \cos\alpha\boldsymbol{i}+\sin\alpha\boldsymbol{j},\quad \boldsymbol{M} = -\sin\alpha\boldsymbol{i}+\cos\alpha\boldsymbol{j}. \tag{11.28}$$

If the fibre strain energy is strictly convex then $f'(\lambda) > 0$ and (11.4) and (11.28a) lead to $\boldsymbol{L}\cdot\nabla\lambda = 0$ for homogenous materials. This result and (11.8a) give $\boldsymbol{L}\cdot\nabla(\boldsymbol{L}\cdot\nabla\phi) = 0$, i.e.

$$\cos^2\alpha\,\partial^2\phi/\partial x^2+2\sin\alpha\cos\alpha\,\partial^2\phi/\partial x\,\partial y+\sin^2\alpha\,\partial^2\phi/\partial y^2 = 0, \tag{11.29}$$

whose general solution is expressible in the form

$$\phi = xa(y-x\tan\alpha)+b(y-x\tan\alpha), \tag{11.30}$$

where $a(\cdot)$ and $b(\cdot)$ are arbitrary. From (11.8*b*) is also follows that

$$\cos\alpha\,\partial\psi/\partial x+\sin\alpha\,\partial\psi/\partial y=0;\quad \psi=c(y-x\tan\alpha), \tag{11.31}$$

for some function $c(\cdot)$. Then in the absence of pressure it is evident from (11.24) that the deformation can be expressed as

$$\boldsymbol{r}=x\boldsymbol{a}(y-x\tan\alpha)+\boldsymbol{b}(y-x\tan\alpha), \tag{11.32}$$

where $\boldsymbol{a}(\cdot)$ and $\boldsymbol{b}(\cdot)$ are vector-valued functions.

As an illustrative example, we consider a rectangular region in the reference plane with boundaries defined by $y=\pm 1$ and $x=0,1$. Suppose the edge $x=0$ is fixed and the edge $x=1$ is collapsed onto a point with coordinates $(x,y,z)=(1,0,h)$, where z measures elevation above the plane. Then the data are

$$\boldsymbol{r}(0,y)=y\boldsymbol{j},\quad \boldsymbol{r}(1,y)=\boldsymbol{i}+h\boldsymbol{k};\quad \boldsymbol{k}=\boldsymbol{i}\times\boldsymbol{j}. \tag{11.33}$$

We take the fibres to be aligned with the coordinate axes so that $\alpha=0$. Then (11.32) and (11.33) give

$$\boldsymbol{b}(y)=y\boldsymbol{j},\quad \boldsymbol{a}(y)=\boldsymbol{i}-y\boldsymbol{j}+h\boldsymbol{k} \tag{11.34}$$

and the deformation is

$$\boldsymbol{r}=x(\boldsymbol{i}+h\boldsymbol{k})+y(1-x)\boldsymbol{j}. \tag{11.35}$$

This provides

$$\left.\begin{aligned}\lambda\boldsymbol{l}&=(\boldsymbol{L}\cdot\nabla)\,\boldsymbol{r}=\boldsymbol{a}(y);\quad \lambda=(1+h^2+y^2)^{\frac{1}{2}},\\ \mu\boldsymbol{m}&=(\boldsymbol{M}\cdot\nabla)\,\boldsymbol{r}=(1-x)\boldsymbol{j};\quad \mu=1-x,\end{aligned}\right\} \tag{11.36}$$

and then (4.8) furnishes the shear angle

$$\sin\gamma=-y/(1+h^2+y^2)^{\frac{1}{2}}. \tag{11.37}$$

We note that $\lambda>1$ and $\mu\in[0,1]$ so that the net is indeed half-slack, in accordance with our assumption.

(*c*) *Surface of revolution: axisymmetry*

Suppose that in the reference configuration the fibres are arranged in an orthogonal network on a surface of revolution, described parametrically by

$$\boldsymbol{x}=r(s)\,\boldsymbol{i}(\theta)+z(s)\,\boldsymbol{k}. \tag{11.38}$$

Here s measures arc length along a meridian, θ is the azimuthal angle, r is the radius, z measures elevation above a base plane with unit normal $\boldsymbol{k}$ and $\boldsymbol{i}$ is a unit vector perpendicular to the axis of revolution and directed radially outward from it. Then the gradient operator is

$$\nabla=\boldsymbol{e}(\partial/\partial s)+r^{-1}\boldsymbol{j}(\partial/\partial\theta);\quad \boldsymbol{e}=r'(s)\,\boldsymbol{i}+z'(s)\,\boldsymbol{k}, \tag{11.39}$$

where $\boldsymbol{e}$ is the unit tangent to a meridian and $\boldsymbol{j}=\boldsymbol{i}'(\theta)$ is tangential to a circle of latitude. Let $\alpha(s)$ be the angle between an $\boldsymbol{L}$-trajectory and a meridian:

$$\boldsymbol{L}=\cos\alpha\boldsymbol{e}+\sin\alpha\boldsymbol{j},\quad \boldsymbol{M}=\sin\alpha\boldsymbol{e}-\cos\alpha\boldsymbol{j}. \tag{11.40}$$

Then (11.4) becomes

$$\partial(fr\cos\alpha)/\partial s+\sin\alpha\,\partial f/\partial\theta=0. \tag{11.41}$$

For axisymmetric stress states ($\partial f/\partial\theta=0$) this gives

$$f=ar^{-1}\sec\alpha;\quad a=\text{const.} \tag{11.42}$$

and therefore
$$\lambda = f^{-1}(ar^{-1}\sec\alpha), \tag{11.43}$$
where $f^{-1}(\cdot)$ is the inverse of $f(\cdot)$, which is uniquely determined for strictly convex fibre strain energies. According to (11.8), (11.39) and (11.40),
$$\cos\alpha\frac{\partial\phi}{\partial s}+r^{-1}\sin\alpha\frac{\partial\phi}{\partial\theta}=\lambda,\quad \cos\alpha\frac{\partial\psi}{\partial s}+r^{-1}\sin\alpha\frac{\partial\psi}{\partial\theta}=0. \tag{11.44}$$
We have some flexibility in constructing the functions ϕ and ψ. If we choose $\partial\psi/\partial\theta = 1$ and let $\psi=\theta$ on a parallel of latitude $s = s_0$, then
$$\psi = \theta - \int_{s_0}^{s} r^{-1}\tan\alpha\,\mathrm{d}\zeta. \tag{11.45}$$
For axisymmetric stress states λ is a function of s only, and thus we can satisfy (11.44a) by taking $\partial\phi/\partial\theta=0$. Setting $\phi(s_0)=0$ we obtain
$$\phi = \int_{s_0}^{s}\lambda\sec\alpha\,\mathrm{d}\zeta. \tag{11.46}$$
The results (11.24), (11.45) and (11.46) provide the general solution for an axisymmetrically stressed net under no normal pressure. It then follows that
$$\left.\begin{aligned}\mu\boldsymbol{m} &= (\boldsymbol{M}\cdot\nabla)\,\boldsymbol{r} = \lambda\tan\alpha\boldsymbol{l} - r^{-1}\sec\alpha[\phi\boldsymbol{l}'(\psi)+\boldsymbol{u}'(\psi)],\\ \mu\sin\gamma &= \mu\boldsymbol{m}\cdot\boldsymbol{l} = \lambda\tan\alpha - r^{-1}\sec\alpha\boldsymbol{l}(\psi)\cdot\boldsymbol{u}'(\psi).\end{aligned}\right\} \tag{11.47}$$
We illustrate this theory with a simple example. Suppose the reference surface is the cylinder $r = \text{const.}$, $-L \leqslant z \leqslant L$ and the fibres are aligned with the meridians and circles of latitude ($\alpha = 0$). Let the ends of the cylinder be fixed onto rigid hoops of radius r. Suppose the hoops are rotated through the relative angle 2τ about the cylinder's axis and displaced along the axis to the relative separation $2l$. For homogeneous materials, (11.43) gives $\lambda = \text{const.}$ and (11.45) and (11.46) give $\psi=\theta$ and $\phi = \lambda z$. Thus
$$\boldsymbol{r} = \lambda z\boldsymbol{l}(\theta)+\boldsymbol{u}(\theta). \tag{11.48}$$
The data are $\boldsymbol{r}(\pm L,\theta) = \pm l\boldsymbol{k}+r\boldsymbol{i}(\theta\pm\tau)$, and therefore
$$\left.\begin{aligned} l\boldsymbol{k}+r\boldsymbol{i}(\theta+\tau) &= \lambda L\boldsymbol{l}(\theta)+\boldsymbol{u}(\theta).\\ -l\boldsymbol{k}+r\boldsymbol{i}(\theta-\tau) &= -\lambda L\boldsymbol{l}(\theta)+\boldsymbol{u}(\theta).\end{aligned}\right\} \tag{11.49}$$
These result in
$$\left.\begin{aligned}&\boldsymbol{u}(\theta) = r\cos\tau\boldsymbol{i}(\theta),\quad \lambda\boldsymbol{l}(\theta) = (l/L)\,\boldsymbol{k}+(r/L)\sin\tau\boldsymbol{j}(\theta)\\ \text{and}\qquad &\lambda^2 = (l/L)^2+(r/L)^2\sin^2\tau.\end{aligned}\right\} \tag{11.50}$$
From (11.47a,b),
$$\left.\begin{aligned}\mu\boldsymbol{m} &= (z/L)\sin\tau\boldsymbol{i}(\theta)-\cos\tau\boldsymbol{j}(\theta);\quad \mu^2 = (z/L)^2\sin^2\tau+\cos^2\tau,\\ \lambda\mu\sin\gamma &= -(r/L)\sin\tau\cos\tau.\end{aligned}\right\} \tag{11.51}$$
Thus $\mu \leqslant 1$ for all $z\in[-L,L]$ and our assumption that the net is half-slack is verified.

The results (11.48) and (11.50) deliver the equation of the deformed surface:
$$\boldsymbol{r} = (z/L)\,l\boldsymbol{k}+r[\cos\tau\boldsymbol{i}(\theta)+(z/L)\sin\tau\boldsymbol{j}(\theta)], \tag{11.52}$$

which can be put into the form

$$\boldsymbol{r} = \rho(z)\,\boldsymbol{i}(\theta+\Gamma(z))+\zeta(z)\,\boldsymbol{k}, \tag{11.53}$$

where ρ is the deformed radius, ζ is the axial position and Γ is the angle of twist. To see this we write

$$\boldsymbol{r} = u(z)\,\boldsymbol{i}(\theta)+v(z)\boldsymbol{j}(\theta)+\zeta(z)\,\boldsymbol{k}, \tag{11.54}$$

where $u = \rho\cos\Gamma$ and $v = \rho\sin\Gamma$. Comparison with (11.52) results in

$$u = r\cos\tau, \quad v = r(z/L)\sin\tau \quad \text{and} \quad \zeta = l(z/L). \tag{11.55}$$

Then the deformed radius and twist angle are given by

$$\rho(z) = r[\cos^2\tau+(z/L)^2\sin^2\tau]^{\frac{1}{2}}, \quad \tan\Gamma(z) = (z/L)\tan\tau. \tag{11.56}$$

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Appendix A

In this appendix we establish the Gauss and Mainardi–Codazzi equations (5.34) and (5.35). For this it is necessary to first find formulae for the various terms in (5.32) and (5.33) as functions of the basic variables λ, μ, γ, κ_l, κ_m and τ. For example, from (2.20 *b*), (4.6) and (5.13 *c, d*) it easily follows that

$$\mu^{\beta\gamma}b_{\alpha\gamma} = \mu L^\beta v_\alpha - \lambda M^\beta u_\alpha, \tag{A 1}$$

where, in consequence of (5.5), we have

$$u_\alpha = \lambda\kappa_l L_\alpha + \mu\tau M_\alpha, \quad v_\alpha = \mu\kappa_m M_\alpha + \lambda\tau L_\alpha. \tag{A 2}$$

According to (2.9 *a*) and (5.11 *a*),

$$p^\alpha = a^{-\frac{1}{2}}\mathrm{e}^{\beta\alpha}l_\beta = J^{-1}\mu^{\beta\alpha}l_\beta. \tag{A 3}$$

Substituting (5.20 *a*) gives

$$Jp^\alpha = \lambda M^\alpha - \mu\sin\gamma L^\alpha. \tag{A 4}$$

From these results we obtain

$$\mu^{\beta\gamma}b_{\alpha\gamma}p^\alpha = J^{-1}\lambda\mu[\mu L^\beta(\kappa_m-\tau\sin\gamma)-\lambda M^\beta(\tau-\kappa_l\sin\gamma)]. \tag{A 5}$$

Combining this result with (A 1) and (A 2 *a*) leads to

$$(\mu^{\beta\gamma}b_{\alpha\gamma}p^\alpha)\,u_\beta = J^{-1}(\lambda\mu)^2(\kappa_l\kappa_m-\tau^2). \tag{A 6}$$

Then in view of (5.8), (5.32 *a*) becomes

$$J\kappa = \mu^{\beta\gamma}x_{\beta,\gamma}, \tag{A 7}$$

where κ is the gaussian curvature. The right-hand side of this result can be written as

$$\mu^{\beta\gamma}x_{\beta,\gamma} = \mu^{\beta\gamma}x_\beta|_\gamma = (\mu^{\beta\gamma}x_\beta)|_\gamma. \tag{A 8}$$

From (2.20 *b*), (4.5) and the definition (5.13 *a*) it follows that

$$\mu^{\beta\gamma}x_\beta = \lambda M^\gamma(\boldsymbol{p}\cdot l^\beta \boldsymbol{l}_{,\beta})-\mu L^\gamma(\boldsymbol{p}\cdot m^\beta\boldsymbol{l}_{,\beta}). \tag{A 9}$$

Then from (5.14a, c) we get

$$\mu^{\beta\gamma} x_\beta = \lambda \eta_l M^\gamma - \mu \phi_l L^\gamma. \quad \text{(A 10)}$$

We can eliminate ϕ_l by using (5.27):

$$\mu^{\beta\gamma} x_\beta = \lambda \eta_l M^\gamma - \mu \eta_m L^\gamma - (J^{-1} \lambda \mu \cos\gamma) M^\alpha L^\gamma \gamma_{,\alpha}. \quad \text{(A 11)}$$

In view of (4.12) it is evident that $J^{-1}\lambda\mu\cos\gamma = \pm 1$ according as $\cos\gamma$ is positive or negative. Then (A 7) becomes

$$J\kappa = (\lambda \eta_l M^\alpha - \mu \eta_m L^\alpha)|_\alpha - (J^{-1}\lambda\mu\cos\gamma)(M^\alpha L^\beta \gamma_{,\alpha})|_\beta. \quad \text{(A 12)}$$

In the last term we can use (2.20b) to obtain the symmetric form

$$(M^\alpha L^\beta \gamma_{,\alpha})|_\beta = \tfrac{1}{2}[(M^\alpha L^\beta + L^\alpha M^\beta)\gamma_{,\alpha}]|_\beta, \quad \text{(A 13)}$$

and thereby arrive at the Gauss equation (5.34). By following a similar procedure we can show that (5.33a) leads to the same result.

Next we derive the Mainardi–Codazzi equations (5.35) from (5.32b) and (5.33b). First we use (4.6) and (5.13a) and (5.14a, c) to write

$$L^\beta x_\beta = \lambda \eta_l, \quad M^\beta x_\beta = \mu \phi_l. \quad \text{(A 14)}$$

These results, combined with (A 5), yield

$$(\mu^{\beta\gamma} b_{\alpha\gamma} p^\alpha) x_\beta = J^{-1}(\lambda\mu)^2[\eta_l(\kappa_m - \tau\sin\gamma) + \phi_l(\kappa_l \sin\gamma - \tau)]. \quad \text{(A 15)}$$

We also have $\mu^{\beta\gamma} u_{\beta,\gamma} = (\mu^{\beta\gamma} u_\beta)|_\gamma$. According to (2.20$b$) and (A 2),

$$\mu^{\beta\gamma} u_\beta = \lambda \kappa_l M^\gamma - \mu\tau L^\gamma. \quad \text{(A 16)}$$

The result (5.35a) follows on substituting the foregoing into (5.32b). Equation (5.35b) follows from (5.33b) similarly.

Pipkin (1984) has derived the Gauss and Mainardi–Codazzi equations for the special case of an initially flat net with inextensible fibres. We can recover his results by taking $\lambda = 1$, $\mu = 1$ identically and $\boldsymbol{L}$, $\boldsymbol{M}$ to be unit vectors in the directions of plane rectangular coordinates x and y. For this case the Gauss equation (5.34) becomes

$$J\kappa = \partial\eta_l/\partial y - \partial\eta_m/\partial x - (J^{-1}\cos\gamma)\,\partial^2\gamma/\partial x\,\partial y, \quad \text{(A 17)}$$

and (5.21a, b) yield

$$\eta_l = (J^{-1}\cos\gamma)\,\partial\gamma/\partial x, \quad \eta_m = -(J^{-1}\cos\gamma)\,\partial\gamma/\partial y. \quad \text{(A 18)}$$

These combine to give

$$\kappa\cos\gamma = \partial^2\gamma/\partial x\,\partial y, \quad \text{(A 19)}$$

in agreement with Pipkin's equation (3.10).

The Mainardi–Codazzi equations (5.35) reduce to

$$\left.\begin{aligned} J(\partial\tau/\partial x - \partial\kappa_l/\partial y) &= \eta_l(\kappa_m - \tau\sin\gamma) + \phi_l(\kappa_l\sin\gamma - \tau), \\ J(\partial\kappa_m/\partial x - \partial\tau/\partial y) &= \phi_m(\kappa_m\sin\gamma - \tau) + \eta_m(\kappa_l - \tau\sin\gamma). \end{aligned}\right\} \quad \text{(A 20)}$$

It follows easily from (5.27) and (A 18) that ϕ_l and ϕ_m vanish identically. Then (A 20a, b) coincide with Pipkin's equations (3.11):

$$\left.\begin{aligned} \cos\gamma(\partial\tau/\partial x - \partial\kappa_l/\partial y) &= (\kappa_m - \tau\sin\gamma)\,\partial\gamma/\partial x, \\ \cos\gamma(\partial\kappa_m/\partial x - \partial\tau/\partial y) &= (\tau\sin\gamma - \kappa_l)\,\partial\gamma/\partial y. \end{aligned}\right\} \quad \text{(A 21)}$$

References

Adkins, J. E. 1956 Finite plane deformation of thin elastic sheets reinforced with inextensible cords. *Phil. Trans. R. Soc. Lond.* A **249**, 125.

Ball, J. M. 1976 On the calculus of variations and sequentially weakly continuous maps. Lecture notes in mathematics no. 564 (Ordinary and partial differential equations), p. 13. Berlin, Heidelberg and New York: Springer-Verlag.

Ball, J. M. 1977*a* Convexity conditions and existence theorems in nonlinear elasticity. *Arch. ration. Mech. Analysis* **63**, 337.

Ball, J. M. 1977*b* Constitutive inequalities and existence theorems in nonlinear elastostatics. Nonlinear Mechanics and Analysis: Heriot–Watt Symposium, vol. 1 (ed. R. J. Knops), p. 187. London, San Francisco and Melbourne: Pitman.

Beatty, M. F. 1987 Topics in finite elasticity: hyperelasticity of rubber, elastomers and biological tissues – with examples. *Appl. Mech. Rev.* **40**, 1699.

Cohen, H. & Wang, C.-C. 1984 On the response and symmetry of elastic and hyperelastic membrane points. *Arch. ration. Mech. Analysis* **85**, 355.

Dacorogna, B. 1982 Quasiconvexity and relaxation of nonconvex problems in the calculus of variations. *J. functional Analysis* **46**, 102.

Dacorogna, B. 1989 *Direct methods in the calculus of variations*. Berlin, Heidelberg and New York: Springer-Verlag.

Ericksen, J. L. 1954 Deformation possible in every isotropic, incompressible, perfectly elastic body. *Z. angew. Math. Phys.* **5**, 446.

Ericksen, J. L. 1955 Deformations possible in every compressible, isotropic, perfectly elastic material. *J. math. Phys.* **34**, 126.

Graves, L. M. 1939 The Weierstrass condition for multiple integral variation problems. *Duke math. J.* **5**, 556.

Green, A. E. & Adkins, J. E. 1970 *Large elastic deformations*, 2nd edn (revised by A. E. Green). Oxford: Clarendon Press.

Green, A. E., Naghdi, P. M. & Wainwright, W. L. 1965 A general theory of a Cosserat surface. *Arch. ration. Mech. Analysis* **20**, 287.

Green, A. E. & Zerna, W. 1968 *Theoretical elasticity*, 2nd edn. Oxford: Clarendon Press.

Green, W. A. & Shi, J. 1989 Plane deformations of membranes formed with elastic cords. (Preprint.)

Hopf, H. 1989 *Differential geometry in the large*. Lecture notes in mathematics no. 1000. Berlin, Heidelberg and New York: Springer-Verlag.

Kohn, R. V. & Strang, G. 1986 Optimal design and relaxation of variational problems I, II and III. *Communs pure appl. Math.* **39**, 113, 139, 353.

Kreyszig, E. 1968 *Introduction to differential geometry and Riemannian geometry*. University of Toronto Press.

Kuznetsov, E. N. 1965 Theory of instantaneously rigid nets. *Appl. Math. Mech.* (*PMM*) **29**, 654.

Kuznetsov, E. N. 1969 *Introduction to the theory of cable systems* (in Russian). Moscow: Publishing House for Construction Literature.

Kuznetsov, E. N. 1982 Axisymmetric static nets. *Int. J. Solids Structures* **18**, 1103.

Kuznetsov, E. N. 1984 Statics and geometry of underconstrained axisymmetric 3-nets. *ASME J. appl. Mech.* **51**, 827.

Morrey, C. B. 1952 Quasi-convexity and the lower semicontinuity of multiple integrals. *Pacific J. Math.* **2**, 25.

Naghdi, P. M. & Tang, P. Y. 1977 Large deformation possible in every isotropic elastic membrane. *Phil. Trans. R. Soc. Lond.* A **287**, 145.

Pipkin, A. C. 1980 Some developments in the theory of inextensible networks. *Q. appl. Math.* **38**, 343.

Pipkin, A. C. 1981 Plane traction problems for inextensible networks. *Q. Jl Mech. appl. Math.* **34**, 415.

Pipkin, A. C. 1984 Equilibrium of Tchebychev nets. *Arch. ration. Mech. Analysis* **85**, 81.

Pipkin, A. C. 1986*a* Continuously distributed wrinkles in fabrics. *Arch. ration. Mech. Analysis* **95**, 93.

Pipkin, A. C. 1986*b* The relaxed energy density for isotropic elastic membranes. *IMA J. appl. Math.* **36**, 85.

Pipkin, A. C. 1989 Elastic materials with two preferred states. (Preprint.)

Pipkin, A. C. & Rogers, T. G. 1987 Infinitesimal plane wrinkling of inextensible networks. *J. Elast.* **17**, 35.

Rivlin, R. S. 1955 Plane strain of a net formed by inextensible cords. *J. ration. Mech. Analysis* **4**, 951.

Rivlin, R. S. 1959 The deformation of a membrane formed by inextensible cords. *Arch. ration. Mech. Analysis* **2**, 447.

Shulikowski, W. I. 1963 *Differential geometry* (in Russian). Moscow: State Publishing House for Physical-Mathematical Literature.

Steigmann, D. J. 1986 Proof of a conjecture in elastic membrane theory. *ASME J. appl. Mech.* **53**, 955.

Steigmann, D. J. 1990 Tension-field theory. *Proc. R. Soc. Lond.* A **429**, 141.

Steigmann, D. J. 1991 A note on pressure potentials. *J. Elast.* (In the press.)

Steigmann, D. J. & Pipkin, A. C. 1989 Axisymmetric tension fields. *Z. angew. Math. Phys.* **40**, 526.

Stoker, J. J. 1964 *Topics in nonlinear elasticity* (notes by R. W. Dickey). Courant Inst. Math. Sci.

Stoker, J. J. 1969 *Differential geometry*. New York, London, Sydney and Toronto: Wiley-Interscience.

Struik, D. J. 1961 *Lectures on classical differential geometry*, 2nd edn. Reading, Massachusetts: Addison-Wesley.

Tchebychev, P. L. 1878 Sur la coupe des vêtements. Assoc. Franc. pour. l'avancement des sci., Congrès des Paris, 154.

Truesdell, C. & Noll, W. 1965 The non-linear field theories of mechanics. S. Flügge's Handbuch der Physik. vol. III/3. Berlin, Heidelberg and New York: Springer-Verlag.

Truesdell, C. & Toupin, R. 1960 The classical field theories. S. Flügge's Handbuch der Physik. vol. III/1, p. 226. Berlin, Göttingen and Heidelberg: Springer-Verlag.

Wang, C. C. & Cross, J. J. 1977 Universal solutions for isotropic elastic membranes. *Arch. ration. Mech. Analysis* **65**, 73.

Weatherburn, C. E. 1961 *Differential geometry*, vol. I. Cambridge University Press.

Yin, W.-L. 1981 Universal solutions for general and area-preserving isotropic elastic membranes. *Arch. ration. Mech. Analysis* **77**, 37.

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